arXiv:1501.04616v1 [math.NA] 19 Jan 2015

DISCRETIZATION OF DIV-CURL SYSTEMS BY WEAK GALERKIN FINITE ELEMENT METHODS ON POLYHEDRAL PARTITIONS

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Abstract. In this paper, the authors devise a new discretization scheme for div-curl systems defined in connected domains with heterogeneous media by using the weak Galerkin finite element method. Two types of boundary value problems are considered in the algorithm development: (1) normal boundary condition, and (2) tangential boundary condition. A new variational formulation is developed for the normal boundary value problem by using the Helmholtz decomposition which avoids the computation of functions in the harmonic fields. Both boundary value problems are reduced to a general saddle-point problem involving the curl and divergence operators, for which the weak Galerkin finite element method is devised and analyzed. The novelty of the technique lies in the discretization of the divergence operator applied to vector fields with heterogeneous media. Error estimates of optimal order are established for the corresponding finite element approximations in various discrete Sobolev norms.

Key words. weak Galerkin, finite element methods, Helmholtz decomposition, weak divergence, weak curl, div-curl systems.

AMS subject classifications. Primary 65N30, 65N12, 65N15; Secondary 35Q60, 35B45.

1. Introduction.

This paper is concerned with new developments of numerical methods for div-curl systems with two types of boundary conditions. The model problem seeks an unknown function $\mathbf{u} = \mathbf{u}(\mathbf{x})$ satisfying

$$\nabla \cdot (\mu \mathbf{u}) = f, \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \times \mathbf{u} = \mathbf{g}, \quad \text{in } \Omega, \tag{1.2}$$

where Ω is an open bounded and connected domain in $\mathbb{R}^3$ with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. The Lebesgue-integrable real-valued function $f = f(\mathbf{x})$ and a vector field $\mathbf{g} = \mathbf{g}(\mathbf{x})$ are given in the domain Ω. Here $\mu = \{\mu_{ij}(\mathbf{x})\}_{3\times 3}$ is a symmetric matrix, uniformly positive definite in Ω and with entries in $L^\infty(\Omega)$. Assume that the boundary Γ has $m+1$ connected components Γ_i such that

$$\Gamma = \bigcup_{i=0}^{m} \Gamma_i,$$

where Γ_0 represents the exterior boundary of Ω, and $\Gamma_i, 1 \le i \le m$, the other connected components of Γ.

The system (1.1)-(1.2) arises in fluid mechanics and electromagnetic field theories. In the fluid mechanics field theory, the coefficient matrix $\mu(\mathbf{x})$ is diagonal where

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diagonal entries are the local mass density. In electrostatics field theory, $\mu(\mathbf{x})$ is the permittivity matrix. In linear magnetic field theory, the function $f(\mathbf{x})$ is zero, $\mathbf{u}$ represents the magnetic field intensity and $\mu(\mathbf{x})$ is the inverse of the magnetic permeability tensor.

We consider two types of boundary conditions for the div-curl system (1.1)-(1.2): the normal boundary condition, and the tangential boundary condition.

The normal boundary condition is concerned with a given flux value for the vector field $\mu\mathbf{u}$ on Γ; i.e.,

$$(\mu\mathbf{u}) \cdot \mathbf{n} = \xi, \qquad \text{on } \Gamma, \tag{1.3}$$

where $\mathbf{n}$ is the unit outward normal direction on Γ.

The tangential boundary condition corresponds to a given value for the tangential component of the vector field $\mathbf{u}$; i.e.,

$$\begin{aligned} \mathbf{u} \times \mathbf{n} &= \boldsymbol{\chi}, && \text{on } \Gamma, \\ \langle \mu\mathbf{u} \cdot \mathbf{n}_i, 1\rangle_{\Gamma_i} &= \beta_i, && i = 1, \ldots, m, \end{aligned} \tag{1.4}$$

where $\mathbf{n}_i$ is the unit outward normal direction on the connected component Γ_i.

The space of harmonic fields is given by

$$\mathbb{H}_\mu(\Omega) = \{\mathbf{v} \in [L^2(\Omega)]^3 : \ \nabla\times\mathbf{v} = 0, \ \nabla \cdot (\mu\mathbf{v}) = 0, \ \mu\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}.$$

The space of harmonic fields is non-trivial when the domain is not simply connected. It is readily seen that the div-curl system (1.1)-(1.2) with the normal boundary condition (1.3) is generally not well-possed, as uniqueness fails (adding a harmonic field to a solution $\mathbf{u}$ still makes a solution). For this reason, we shall look for a solution $\mathbf{u}$ that is orthogonal to the space of harmonic fields in the μ-weighted L^2 norm. Specifically, the complete problem reads: given $\mathbf{g} \in [L^2(\Omega)]^3$ with $\nabla \cdot \mathbf{g} = 0$ in Ω, find $\mathbf{u}$ such that

$$\begin{cases} \nabla \cdot (\mu\mathbf{u}) = f, & \text{in } \Omega, \\ \nabla \times \mathbf{u} = \mathbf{g}, & \text{in } \Omega, \\ \mu\mathbf{u} \cdot \mathbf{n} = \xi, & \text{on } \Gamma, \\ \displaystyle\int_\Omega \mu\mathbf{u} \cdot \boldsymbol{\eta} = 0, & \forall\, \boldsymbol{\eta} \in \mathbb{H}_\mu(\Omega). \end{cases} \tag{1.5}$$

The problem (1.5) has a solution [28], and uniqueness is straightforward. It is also well-known that the div-curl system (1.1)-(1.2) with the tangential boundary condition (1.4) has one and only one solution.

There have been many numerical methods for approximating div-curl systems. In [22], Nicolaides proposed and analyzed a control volume method directly for planar div-curl problems. In [23], Nicolaides and Wu presented a special co-finite volume method for div-curl problems in three dimension, which was based on a system of two orthogonal grids like the classical Voronoi-Delaunay mesh pair. Bossavit [3] proposed a classical numerical method for solving the magnetostatic problem by introducing a scalar or vector potential. In [12], a discrete duality finite volume method was presented for div-curl problems on almost arbitrary polygonal meshes. In [4], Bramble and Pasciak proposed and analyzed a direct numerical scheme under a very weak

formulation where the solution space was $[L^2(\Omega)]^3$. In [10], a mixed finite element method was analyzed for div-curl systems in simply connected axisymmetric domains. In [26], a numerical algorithm was designed for constructing a finite element basis for the first de Rham cohomology group of the computational domain, which was further used for a numerical approximation of the magnetostatic problem. In [7] and [17], the mimetic finite difference method was applied to the 3D magnetostatic problems on general polyhedral meshes.

Recently, weak Galerkin (WG) finite element methods have emerged as a new numerical technique for approximating the solutions of partial differential equations. The WG method was first introduced in [30, 32] for the second order elliptic problem and was further developed in [30, 19, 31, 29] with other applications. Two basic principles for the WG finite element method are: (1) the differential operators (e.g., gradient, Laplacian, Hessian, curl, divergence etc.) are interpreted and approximated as distributions over a set of generalized functions, and (2) proper stabilizations are employed to enforce necessary weak continuities for approximating functions in the correct topology. It has been demonstrated that the WG method is highly flexible and robust as a numerical tool that makes use of discontinuous piecewise polynomials on polygonal or polyhedral finite element partitions.

The goal of this paper is to present a new discretization scheme for the div-curl system (1.1)-(1.2) in any connected domain with heterogeneous media by using the weak Galerkin finite element approach. In particular, for the normal boundary value problem (1.5), a new variational formulation is developed by using a Helmholtz decomposition which avoids the computation of any harmonic fields $\mathbf{v} \in \mathbb{H}_\mu(\Omega)$. The resulting formulation is a special case of the model problem (5.1) detailed in Section 5. The div-curl system with the tangential boundary condition (1.4) is also formulated as a special case of the model problem (5.1). Therefore, our attention is focused on the development of weak Galerkin finite element methods for (5.1). It is readily seen that the main difficulty in numerical methods for (5.1) lies in the term $\nabla \cdot (\mu \mathbf{u})$ which requires the continuity of $\mu \mathbf{u}$ in the normal direction of any interface, particularly the interface of any two polyhedral elements. The weak Galerkin finite element method offers an ideal solution, as the continuity can be relaxed by a weak continuity implemented through a carefully chosen stabilizer.

The paper is organized as follows. In Section 2, we introduce some commonly used notations. In Section 3, we derive a formulation for the div-curl problem (1.1)-(1.2) with normal boundary condition (1.3) by using Helmholtz decomposition. Section 4 is devoted to a discussion of the div-curl system with tangential boundary condition. In Section 5, we discuss a model problem that is central to the solution of the div-curl system with both the normal and tangential boundary conditions. In Section 6, we introduce some discrete weak differential operators which are necessary for the development of weak Galerkin finite element methods in Section 7. Section 8 is devoted to a discussion of existence and uniqueness for the solution of the weak Galerkin discretizations. In Section 9, we derive some error equations. An *inf-sup* condition is established in Section 10. Finally in Section 11, we derive some optimal order error estimates for the WG finite element approximations.

2. Notations and Preliminaries. Throughout the paper, we will follow the usual notation for Sobolev spaces and norms [9]. For any open bounded domain $D \subset \mathbb{R}^3$ with Lipschitz continuous boundary, we use $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ to denote the

norm and seminorm in the Sobolev space $H^s(D)$ for any $s \geq 0$, respectively. The inner product in $H^s(D)$ is denoted by $(\cdot,\cdot)_{s,D}$. The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively.

Let $\mu = \{\mu_{ij}\}_{3\times 3}$ be a symmetric matrix, uniformly positive definite in D and with entries in $L^\infty(D)$. Introduce the following Sobolev space

$$H(\mathrm{div}_\mu; D) = \{\mathbf{v} \in [L^2(D)]^3 :\ \nabla\cdot(\mu\mathbf{v}) \in L^2(D)\},$$

with norm given by

$$\|\mathbf{v}\|_{H(\mathrm{div}_\mu;D)} = (\|\mathbf{v}\|_D^2 + \|\nabla\cdot(\mu\mathbf{v})\|_D^2)^{\frac{1}{2}},$$

where $\nabla\cdot(\mu\mathbf{v})$ is the divergence of $\mu\mathbf{v}$. Any $\mathbf{v} \in H(\mathrm{div}_\mu; D)$ can be assigned a trace for the normal component of $\mu\mathbf{v}$ on the boundary. The subspace with vanishing trace in the normal component is denoted by

$$H_0(\mathrm{div}_\mu; D) = \{\mathbf{v} \in H(\mathrm{div}_\mu; D) :\ (\mu\mathbf{v})\cdot\mathbf{n}|_{\partial D} = 0\}.$$

Denote the subspace of $H_0(\mathrm{div}_\mu; D)$ with divergence-free vectors by

$$\mathbb{F}_\mu(D) = \{\mathbf{v} \in H(\mathrm{div}_\mu; D) :\ \nabla\cdot(\mu\mathbf{v}) = 0\}.$$

When $\mu = I$ is the identity matrix, the spaces $H(\mathrm{div}_\mu; D)$, $H_0(\mathrm{div}_\mu; D)$, and $\mathbb{F}_\mu(D)$ shall be denoted as $H(\mathrm{div}; D)$, $H_0(\mathrm{div}; D)$, and $\mathbb{F}(D)$, respectively.

Denote by $H(\mathrm{curl}; D)$ the following Sobolev space

$$H(\mathrm{curl}; D) = \{\mathbf{v} : \mathbf{v} \in [L^2(D)]^3, \nabla\times\mathbf{v} \in [L^2(D)]^3\}$$

with norm defined by

$$\|\mathbf{v}\|_{H(\mathrm{curl};D)} = (\|\mathbf{v}\|_D^2 + \|\nabla\times\mathbf{v}\|_D^2)^{\frac{1}{2}},$$

where $\nabla\times\mathbf{v}$ is the curl of $\mathbf{v}$. Any $\mathbf{v} \in H(\mathrm{curl}; D)$ can be assigned a trace for its tangential component on the boundary. The subspace of $H(\mathrm{curl}; D)$ with vanishing trace in the tangential component is denoted by

$$H_0(\mathrm{curl}; D) = \{\mathbf{v} \in H(\mathrm{curl}; D) :\ \mathbf{v}\times\mathbf{n}|_{\partial D} = 0\}.$$

When $D = \Omega$, we shall drop the subscript D in the norm and inner product notation.

For simplicity of notation, throughout the paper, we use "$\lesssim$ " to denote "less than or equal to up to a general constant independent of the mesh size or functions appearing in the inequality".

3. The div-curl System with Normal Boundary Condition. The goal of this section is to derive a suitable variational formulation for the problem (1.5). Denote by $H^0(\mathrm{curl}; \Omega) = \{\mathbf{v} \in [L^2(\Omega)]^3 :\ \nabla\times\mathbf{v} = 0\}$ the space of curl-free fields. It is well-known that any vector field $\mathbf{v} \in H^0(\mathrm{curl}; \Omega)$ can be written as (see, e.g., [27])

$$\mathbf{v} = \nabla\phi + \boldsymbol{\eta}, \tag{3.1}$$

where $\phi \in H^1(\Omega)$ and $\boldsymbol{\eta} \in \mathbb{H}_\mu(\Omega)$. It follows that

$$(\nabla\phi, \mu\boldsymbol{\eta}) = -(\phi, \nabla\cdot(\mu\boldsymbol{\eta})) + \langle\phi, \mu\boldsymbol{\eta}\cdot\mathbf{n}\rangle_\Gamma = 0.$$

The decomposition (3.1) is thus orthogonal in the μ-weighted L^2 norm in $H^0(\mathrm{curl}; \Omega)$.

3.1. Helmholtz decomposition. Denote by $H^1_{0c}(\Omega)$ the set of functions in $H^1(\Omega)$ with vanishing value on Γ_0 and constant values on other connected components of the boundary; i.e.,

$$H^1_{0c}(\Omega) = \{\phi \in H^1(\Omega) : \ \phi|_{\Gamma_0} = 0, \ \phi|_{\Gamma_i} = c, \ i = 1, \ldots, m\}.$$

Next, introduce a Sobolev space

$$\mathbb{Y}_\mu(\Omega) = \{\mathbf{v} \in H_0(\text{curl}; \Omega) : \ \langle \mu\mathbf{v} \cdot \mathbf{n}_i, 1\rangle_{\Gamma_i} = 0, \ i = 1, \ldots, m\}.$$

The following Helmholtz decomposition holds the key to the derivation of a suitable variational form for the problem (1.5).

THEOREM 3.1. *For any vector-valued function $\boldsymbol{u} \in [L^2(\Omega)]^3$, there exists a unique $\boldsymbol{\psi} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega)$, $\phi \in H^1(\Omega)/\mathbb{R}$, and $\boldsymbol{\eta} \in \mathbb{H}_\mu(\Omega)$ such that*

$$\boldsymbol{u} = \mu^{-1}\nabla \times \boldsymbol{\psi} + \nabla\phi + \boldsymbol{\eta}. \tag{3.2}$$

Moreover, the following estimate holds true

$$\|\boldsymbol{\psi}\|_{H(\text{curl};\Omega)} + \|\nabla\phi\|_0 \lesssim (\kappa\boldsymbol{u}, \boldsymbol{u})^{\frac{1}{2}}. \tag{3.3}$$

Proof. Consider the problem of seeking $\boldsymbol{\psi} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega)$ such that

$$(\mu^{-1}\nabla \times \boldsymbol{\psi}, \nabla \times \boldsymbol{\varphi}) = (\mathbf{u}, \nabla \times \boldsymbol{\varphi}), \qquad \forall\ \boldsymbol{\varphi} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega). \tag{3.4}$$

Denote by

$$a(\boldsymbol{\psi}, \boldsymbol{\varphi}) := (\mu^{-1}\nabla \times \boldsymbol{\psi}, \nabla \times \boldsymbol{\varphi})$$

the bilinear form defined on $\mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega)$. We claim that $a(\cdot, \cdot)$ is coercive with respect to the $H(\text{curl}; \Omega)$-norm. To this end, it suffices to derive the following estimate

$$\|\mathbf{v}\|_0 \lesssim \|\nabla \times \mathbf{v}\|_0, \qquad \forall \mathbf{v} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega). \tag{3.5}$$

In fact, for any $\mathbf{v} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega)$, from Theorem 3.4 of [13], there exists a vector potential function $\boldsymbol{\omega} \in [H^1(\Omega)]^3$ such that

$$\mu\mathbf{v} = \nabla \times \boldsymbol{\omega}, \ \nabla \cdot \boldsymbol{\omega} = 0, \ \|\boldsymbol{\omega}\|_1 \lesssim (\mu\mathbf{v}, \mathbf{v})^{\frac{1}{2}}. \tag{3.6}$$

Using the integration by parts and the condition $\mathbf{v} \times \mathbf{n} = 0$ on Γ, we have

$$(\mu\mathbf{v}, \mathbf{v}) = (\nabla \times \boldsymbol{\omega}, \mathbf{v}) = (\boldsymbol{\omega}, \nabla \times \mathbf{v}).$$

It follows from the Cauchy-Schwarz inequality and (3.6) that

$$(\mu\mathbf{v}, \mathbf{v}) \le \|\boldsymbol{\omega}\|_0 \ \|\nabla \times \mathbf{v}\|_0 \lesssim (\mu\mathbf{v}, \mathbf{v})^{\frac{1}{2}} \ \|\nabla \times \mathbf{v}\|_0,$$

which implies (3.5).

Now from the Lax-Milgram Theorem, there exists a unique $\boldsymbol{\psi} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega)$ satisfying the equation (3.4) such that

$$\|\boldsymbol{\psi}\|_{H(\text{curl};\Omega)} \lesssim \|\mathbf{u}\|_0 \lesssim (\mu\mathbf{u}, \mathbf{u}).$$

It is easy to see that $\mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega)$ is equivalent to the following quotient space:

$$H_0(\text{curl};\Omega)/(\nabla H^1_{0c}(\Omega)) = \{\mathbf{v} \in H_0(\text{curl};\Omega) : \ (\mu\mathbf{v}, \nabla\phi) = 0, \ \forall \phi \in H^1_{0c}(\Omega)\}.$$

Thus, by using a Lagrange multiplier $p \in H^1_{0c}(\Omega)$, the problem (3.4) can be reformulated as follows: Find $\boldsymbol{\psi} \in H_0(\text{curl};\Omega)$ and $p \in H^1_{0c}(\Omega)$ such that

$$\begin{aligned} (\mu^{-1}\nabla\times\boldsymbol{\psi}, \nabla\times\boldsymbol{\varphi}) + (\mu\nabla p, \boldsymbol{\varphi}) &= (\mathbf{u}, \nabla\times\boldsymbol{\varphi}), \quad && \forall\ \boldsymbol{\varphi} \in H_0(\text{curl};\Omega), \\ (\boldsymbol{\psi}, \mu\nabla s) &= 0, && \forall\ s \in H^1_{0c}(\Omega). \end{aligned} \tag{3.7}$$

It follows from the first equation of (3.7) that

$$\nabla\times(\mathbf{u} - \mu^{-1}\nabla\times\boldsymbol{\psi}) - \mu\nabla p = 0.$$

Since $p \in H^1_{0c}(\Omega)$, then the two terms on the left-hand side of the above equation are orthogonal in the μ^{-1}-weighted $L^2(\Omega)$ norm. Thus,

$$\nabla\times(\mathbf{u} - \mu^{-1}\nabla\times\boldsymbol{\psi}) = 0 \Longrightarrow \ \mathbf{u} - \mu^{-1}\nabla\times\boldsymbol{\psi} \in H^0(\text{curl};\Omega).$$

Thus, there exist unique $\phi \in H^1(\Omega)/\mathbb{R}$ and $\boldsymbol{\eta} \in \mathbb{H}_\mu(\Omega)$ such that

$$\mathbf{u} - \kappa^{-1}\nabla\times\boldsymbol{\psi} = \nabla\phi + \boldsymbol{\eta},$$

which completes the proof of the theorem. □

3.2. A variational formulation. Assume that the div-curl problem (1.1)-(1.2) with boundary condition (1.3) has a solution. Integrating (1.1) over the domain Ω and from the integration by parts, we have

$$(f, 1) = (\nabla\cdot(\mu\mathbf{u}), 1) = \langle(\mu\mathbf{u})\cdot\mathbf{n}, 1\rangle_\Gamma.$$

Using the boundary condition (1.3), we arrive at the following compatibility condition

$$(f, 1) = \langle\xi, 1\rangle_\Gamma. \tag{3.8}$$

In addition, taking the divergence to the equation (1.2), we obtain a second compatibility condition

$$\nabla\cdot\mathbf{g} = 0 \qquad \text{in } \Omega. \tag{3.9}$$

Lemma 3.2. *Let $\boldsymbol{u}$ be a solution of the div-curl system (1.1)-(1.2) with the boundary condition (1.3). Then, $\boldsymbol{u}$ can be decomposed as follows*

$$\boldsymbol{u} = \mu^{-1}\nabla\times\boldsymbol{\psi} + \nabla\phi + \boldsymbol{\eta}, \tag{3.10}$$

where $\boldsymbol{\psi} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega)$ is the unique solution of the following equation

$$(\mu^{-1}\nabla\times\boldsymbol{\psi}, \nabla\times\boldsymbol{\zeta}) = (\mathbf{g}, \boldsymbol{\zeta}), \qquad \forall\ \boldsymbol{\zeta} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega), \tag{3.11}$$

and $\phi \in H^1(\Omega)/\mathbb{R}$ satisfies

$$(\mu\nabla\phi, \nabla v) = \langle\xi, v\rangle - (f, v), \qquad \forall\ v \in H^1(\Omega)/\mathbb{R}. \tag{3.12}$$

Proof. Using the Helmholtz decomposition (3.2) in Theorem 3.1, there exist unique $\boldsymbol{\psi} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega)$, $\phi \in H^1(\Omega)/\mathbb{R}$, and $\boldsymbol{\eta} \in \mathbb{H}_\mu(\Omega)$ such that

$$\mathbf{u} = \mu^{-1}\nabla \times \boldsymbol{\psi} + \nabla\phi + \boldsymbol{\eta}. \tag{3.13}$$

For any $\boldsymbol{\zeta} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega)$,

$$(\nabla\phi, \nabla\times\boldsymbol{\zeta}) = (\nabla\times\nabla\phi, \boldsymbol{\zeta}) - \langle \nabla\phi, \boldsymbol{\zeta} \times \mathbf{n}\rangle_\Gamma = 0.$$

Thus, by testing both sides of (3.13) with $\nabla\times\boldsymbol{\zeta}$, we obtain

$$(\mathbf{u} - \boldsymbol{\eta}, \nabla\times\boldsymbol{\zeta}) = (\mu^{-1}\nabla\times\boldsymbol{\psi}, \nabla\times\boldsymbol{\zeta}). \tag{3.14}$$

From the equation (1.2) and the fact that $\nabla\times\boldsymbol{\eta} = 0$ and $\boldsymbol{\zeta} \times \mathbf{n} = 0$ on Γ, we obtain

$$(\mathbf{u} - \boldsymbol{\eta}, \nabla\times\boldsymbol{\zeta}) = (\nabla\times\mathbf{u}, \boldsymbol{\zeta}) = (\mathbf{g}, \boldsymbol{\zeta}).$$

Substituting the above into (3.14) yields

$$(\mu^{-1}\nabla\times\boldsymbol{\psi}, \nabla\times\boldsymbol{\zeta}) = (\mathbf{g}, \boldsymbol{\zeta}), \qquad \forall\ \boldsymbol{\zeta} \in \mathbb{Y}_\mu(\Omega) \cap \mathbb{F}_\mu(\Omega), \tag{3.15}$$

which verifies (3.11).

Next, we test (3.13) against any $\mu\nabla\varphi$ to obtain

$$(\mathbf{u}, \mu\nabla\varphi) = (\mu^{-1}\nabla\times\boldsymbol{\psi}, \mu\nabla\varphi) + (\mu\nabla\phi, \nabla\varphi) + (\boldsymbol{\eta}, \mu\nabla\varphi), \qquad \forall\varphi \in H^1(\Omega). \tag{3.16}$$

Using the integration by parts,

$$(\mathbf{u}, \mu\nabla\varphi) = -(\nabla \cdot (\mu\mathbf{u}), \varphi) + \langle (\mu\mathbf{u}) \cdot \mathbf{n}, \varphi\rangle_\Gamma.$$

Thus, from the equation (1.1) and the boundary condition (1.3), we have

$$(\mathbf{u}, \mu\nabla\varphi) = -(\mathbf{f}, \varphi) + \langle \xi, \varphi\rangle_\Gamma. \tag{3.17}$$

Similarly, from the integration by parts and the fact that $\boldsymbol{\eta} \in \mathbb{H}_\mu(\Omega)$,

$$(\boldsymbol{\eta}, \mu\nabla\varphi) = -(\nabla \cdot (\mu\boldsymbol{\eta}), \varphi) + \langle (\mu\boldsymbol{\eta}) \cdot \mathbf{n}, \varphi\rangle_\Gamma = 0. \tag{3.18}$$

Since $\boldsymbol{\psi} \in H_0(\mathrm{curl}; \Omega)$, then

$$(\nabla\times\boldsymbol{\psi}, \nabla\varphi) = 0. \tag{3.19}$$

Substituting (3.17)-(3.19) into (3.16) gives rise to

$$(\mu\nabla\phi, \nabla\varphi) = \langle \xi, \varphi\rangle_\Gamma - (\mathbf{f}, \varphi), \qquad \forall\varphi \in H^1(\Omega). \tag{3.20}$$

The above problem has a unique solution in the quotient space $H^1(\Omega)/\mathbb{R}$ due to the compatibility condition (3.8). This completes the proof. □

By using a Lagrange multiplier p, the problem (3.11) can be re-formulated as a saddle point problem that seeks $\boldsymbol{\psi} \in \mathbb{Y}_\mu(\Omega) \cap H(\mathrm{div}_\mu; \Omega)$ and $p \in L^2(\Omega)$ satisfying

$$\begin{aligned} (\mu^{-1}\nabla\times\boldsymbol{\psi}, \nabla\times\boldsymbol{\zeta}) + (\nabla \cdot (\mu\boldsymbol{\zeta}), p) &= (\mathbf{g}, \boldsymbol{\zeta}), \qquad &&\forall\ \boldsymbol{\zeta} \in \mathbb{Y}_\mu(\Omega) \cap H(\mathrm{div}_\mu; \Omega), \\ (\nabla \cdot (\mu\boldsymbol{\psi}), w) &= 0, &&\forall\ w \in L^2(\Omega). \end{aligned} \tag{3.21}$$

Going back to the well-posed problem (1.5), it is easily seen that the solution of (1.5) also admits the decomposition (3.10). Thus, using $\boldsymbol{\psi}\times\mathbf{n}=0$ and $\mu\boldsymbol{\eta}\cdot\mathbf{n}=0$ on Γ, we obtain

$$
\begin{array}{rcll}
0 & = & (\mu\mathbf{u},\boldsymbol{\eta}) & \text{by last condition in (1.5)}\\
& = & (\nabla\times\boldsymbol{\psi},\boldsymbol{\eta})+(\mu\nabla\phi,\boldsymbol{\eta})+(\mu\boldsymbol{\eta},\boldsymbol{\eta}) & \text{by the decomposition (3.10)}\\
& = & (\boldsymbol{\psi},\nabla\times\boldsymbol{\eta})-(\phi,\nabla\cdot(\mu\boldsymbol{\eta}))+(\mu\boldsymbol{\eta},\boldsymbol{\eta}) & \text{by integration by parts}\\
& = & (\mu\boldsymbol{\eta},\boldsymbol{\eta}). & \text{by } \boldsymbol{\eta}\in\mathbb{H}_\mu(\Omega)
\end{array}
$$

It follows that $\boldsymbol{\eta}\equiv 0$. The result can be summarized as follows.

THEOREM 3.3. *Let $\boldsymbol{u}$ be the solution of the div-curl system (1.5). Then, $\boldsymbol{u}$ can be represented*

$$\boldsymbol{u}=\nabla\times\boldsymbol{\psi}+\nabla\phi, \tag{3.22}$$

where $\boldsymbol{\psi}\in\mathbb{Y}_\mu(\Omega)\cap\mathbb{F}_\mu(\Omega)$ is the unique solution of the system of equations (3.21), and $\phi\in H^1(\Omega)$ is determined by the equation (3.12).

Theorem 3.3 indicates that a suitable variational formation for the div-curl system (1.5) is given by (3.21) and (3.12), which are independently defined and each has one and only one solution in the corresponding Sobolev space. The problem (3.12) is a standard second order elliptic equation for which many existing numerical methods can be applied. But the problem (3.21) requires a study in numerical methods, which is the central topic of this paper.

4. The div-curl System with Tangential Boundary Condition. The goal of this section is to derive a variational formulation for the div-curl system with the tangential boundary condition (1.4). Recall that the complete problem reads: given $\mathbf{g}\in[L^2(\Omega)]^3$ with $\nabla\cdot\mathbf{g}=0$ in Ω, find the vector-valued function $\mathbf{u}$ such that

$$
\left\{
\begin{array}{rll}
\nabla\cdot(\mu\mathbf{u})=f, & & \text{in } \Omega,\\
\nabla\times\mathbf{u}=\mathbf{g}, & & \text{in } \Omega,\\
\mathbf{u}\times\mathbf{n}=\boldsymbol{\chi}, & & \text{on } \Gamma,\\
\langle\mu\mathbf{u}\cdot\mathbf{n}_i,1\rangle_{\Gamma_i}=\beta_i, & & i=1,\ldots,m.
\end{array}
\right. \tag{4.1}
$$

The problem (4.1) can be interpreted as a constraint minimization problem with an object function given by

$$J(\mathbf{v})=\frac{1}{2}\|\nabla\times\mathbf{v}-\mathbf{g}\|_0^2,$$

subject to the following constraints

$$
\left\{
\begin{array}{rll}
\nabla\cdot(\mu\mathbf{v})=f, & & \text{in } \Omega,\\
\mathbf{v}\times\mathbf{n}=\boldsymbol{\chi}, & & \text{on } \Gamma,\\
\langle\mu\mathbf{v}\cdot\mathbf{n}_i,1\rangle_{\Gamma_i}=\beta_i, & & i=1,\ldots,m.
\end{array}
\right. \tag{4.2}
$$

By introducing a Lagrange multiplier p, the corresponding variational problem seeks $\mathbf{u}\in H(\mathrm{curl};\Omega)\cap H(\mathrm{div}_\mu;\Omega)$ and $p\in L^2(\Omega)$ with $\mathbf{u}\times\mathbf{n}=\boldsymbol{\chi}$ on Γ and $\langle\mu\mathbf{u}\cdot\mathbf{n}_i,1\rangle_{\Gamma_i}=\beta_i$ for $i=1,\ldots,m$ such that

$$
\begin{array}{rl}
(\nabla\times\mathbf{u},\nabla\times\boldsymbol{\zeta})+(\nabla\cdot(\mu\boldsymbol{\zeta}),p)=(\nabla\times\mathbf{g},\boldsymbol{\zeta}), & \forall\,\boldsymbol{\zeta}\in\mathbb{Y}_\mu(\Omega)\cap H(\mathrm{div}_\mu;\Omega),\\
(\nabla\cdot(\mu\mathbf{u}),w)=(f,w), & \forall\, w\in L^2(\Omega).
\end{array} \tag{4.3}
$$

The problem (4.3) is the desired variational form for the div-curl system with tangential boundary condition (1.4). The structure of this variational problem is essentially the same as that of (3.21) arising from the normal boundary condition.

LEMMA 4.1. *The solution to the variational problem (4.3) is unique for any connected domain* Ω.

Proof. It suffices to show that all the solutions corresponding to the one with homogeneous data are trivial. To this end, let $(\mathbf{u}; p)$ be a solution of (4.3) with homogeneous data, then

$$\nabla\times\mathbf{u} = 0, \quad \text{in } \Omega, \tag{4.4}$$

$$\nabla\cdot(\mu\mathbf{u}) = 0, \quad \text{in } \Omega, \tag{4.5}$$

$$\langle \mu\mathbf{u}\cdot\mathbf{n}_i, 1\rangle_{\Gamma_i} = 0, \quad i = 1, \ldots, m, \tag{4.6}$$

$$\mathbf{u}\times\mathbf{n} = 0, \quad \text{on } \partial\Omega. \tag{4.7}$$

From (4.5) and (4.6), the conditions of Theorem 3.4 in [13] are satisfied for the vector-valued function $\mu\mathbf{u}$. Thus, there exists a vector potential $\boldsymbol{\phi} \in [H^1(\Omega)]^3$ such that $\mu\mathbf{u} = \nabla\times\boldsymbol{\phi}$. It follows that

$$\begin{aligned}
(\mu\mathbf{u}, \mathbf{u}) &= (\nabla\times\boldsymbol{\phi}, \mathbf{u}) \\
&= (\boldsymbol{\phi}, \nabla\times\mathbf{u}) + \langle \mathbf{n}\times\boldsymbol{\phi}, \mathbf{u}\rangle_\Gamma && \text{by integration by parts} \\
&= \langle \phi, \mathbf{u}\times\mathbf{n}\rangle_\Gamma && \text{by (4.4) and triple product property} \\
&= 0, && \text{by (4.7)}
\end{aligned}$$

which implies $\mathbf{u} \equiv 0$. Consequently, one has the following equation

$$(\nabla\cdot(\mu\boldsymbol{\zeta}), p) = 0, \qquad \forall \boldsymbol{\zeta} \in \mathbb{Y}_\mu(\Omega)\cap H(\mathrm{div}_\mu; \Omega),$$

which leads to $p = 0$ by selecting a particular vector field $\boldsymbol{\zeta} \in \mathbb{Y}_\mu(\Omega)\cap H(\mathrm{div}_\mu; \Omega)$ such that $\nabla\cdot(\mu\boldsymbol{\zeta}) = p$. Such a vector field is given by $\boldsymbol{\zeta} = \nabla w$, where w is the solution of the following equation

$$\nabla\cdot(\mu\nabla w) = p, \quad w|_{\Gamma_0} = 0, \quad w|_{\Gamma_i} = \gamma_i, \; i = 1, \ldots, m.$$

Here, $\{\gamma_i\}$ is a set of real numbers which can be tuned so that $\langle \mu\nabla w\cdot\mathbf{n}_i, 1\rangle_{\Gamma_i} = 0$. ∎

5. A Model Problem. It is readily seen from Subsections 3.2 and Section 4 that the core problem of study for the div-curl system with either the normal or the tangential boundary conditions is one structured in the form of (3.21) and (4.3). Thus, we shall consider a general problem in the form of (3.21) and (4.3) as follows. Assume the following data are given: $\mathbf{g} \in [L^2(\Omega)]^3$, $f \in L^2(\Omega)$, $\xi \in [H^{-\frac{1}{2}}(\Gamma)]^3$, a symmetric and positive definite matrix $\kappa = (\kappa_{ij}(\mathbf{x}))_{3\times3}$ in the domain Ω, and a set of real numbers $\beta_i, i = 1, \ldots, m$.

PROBLEM 1. *Find* $\boldsymbol{u} \in H(\mathrm{curl}; \Omega)\cap H(\mathrm{div}_\mu; \Omega)$ *and* $p \in L^2(\Omega)$ *such that*

$$\left\{\begin{aligned}
(\kappa\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v}) + (\nabla\cdot(\mu\boldsymbol{v}), p) &= (\mathbf{g}, \boldsymbol{v}), && \forall\, \boldsymbol{v} \in \mathbb{Y}_\mu(\Omega)\cap H(\mathrm{div}_\mu; \Omega), \\
(\nabla\cdot(\mu\boldsymbol{u}), w) &= (f, w), && \forall\, w \in L^2(\Omega), \\
\langle \mu\boldsymbol{u}\cdot\mathbf{n}_i, 1\rangle_{\Gamma_i} &= \beta_i, && i = 1, \ldots, m, \\
\boldsymbol{u}\times\mathbf{n} &= \xi, && \text{on } \Gamma.
\end{aligned}\right. \tag{5.1}$$

Here $\mathbf{n}$ *is the unit outward normal direction to the boundary* Γ.

The constraint of $\langle \mu\mathbf{v}\cdot\mathbf{n}_i, 1\rangle_{\Gamma_i} = 0$ for the test space $\mathbb{Y}_\mu(\Omega)\cap H(\mathrm{div}_\mu;\Omega)$ is cumbersome in the design of numerical methods for the problem (5.1). One possible remedy is to relax this constraint by using a Lagrange multiplier denoted by $\lambda = (\lambda_1,\ldots,\lambda_m)$. The corresponding weak formulation seeks $\mathbf{u}\in H(\mathrm{curl};\Omega)\cap H(\mathrm{div}_\mu;\Omega)$, $p\in L^2(\Omega)$, and $\lambda\in\mathbb{R}^m$ such that $\mathbf{u}\times\mathbf{n}=\xi$ on Γ and

$$
(5.2)\qquad \left\{ \begin{aligned} &(\kappa\nabla\times\mathbf{u},\nabla\times\mathbf{v}) + (\nabla\cdot(\mu\mathbf{v}),p) + \sum_{i=1}^m\langle\mu\mathbf{v}\cdot\mathbf{n}_i,\lambda_i\rangle_{\Gamma_i} = (\mathbf{g},\mathbf{v}),\\ &(\nabla\cdot(\mu\mathbf{u}),w) + \sum_{i=1}^m\langle\mu\mathbf{u}\cdot\mathbf{n}_i,s_i\rangle_{\Gamma_i} = (f,w) + \sum_{i=1}^m\beta_i s_i, \end{aligned}\right.
$$

for all $\mathbf{v}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div}_\mu;\Omega)$, $w\in L^2(\Omega)$, and $s\in\mathbb{R}^m$. It is not hard to see that the problems (5.2) and (5.1) are equivalent to each other.

For simplicity of analysis, throughout the paper, we assume that κ and μ are piecewise constant, symmetric and positive definite matrices on the domain Ω with respect to any finite element partitions to be specified in forthcoming sections.

6. Weak Differential Operators. The model problem (5.1) is formulated with two principle differential operators: divergence and curl. This section shall introduce the notion of weak divergence operator for vector-valued functions of the form $\mu\mathbf{v}$. For completeness, we also review the definition for the weak curl operator. These weak differential operators shall be discretized by using polynomials, which leads to discretizations for the model problem (5.1).

Let $K\subset\Omega$ be any open bounded domain with boundary ∂K. Denote by $\mathbf{n}$ the unit outward normal direction on ∂K. Let the space of weak vector-valued functions in K be given by

$$
V(K) = \{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}:\ \mathbf{v}_0\in[L^2(K)]^3,\ \mathbf{v}_b\in[L^2(\partial K)]^3\},
$$

where $\mathbf{v}_0$ represents the value of $\mathbf{v}$ in the interior of K, and $\mathbf{v}_b$ represents certain information of $\mathbf{v}$ on the boundary ∂K. There are two pieces of information of $\mathbf{v}$ on ∂K that are necessary for defining the variational formulation (5.1): (1) the tangential component $\mathbf{n}\times(\mathbf{v}\times\mathbf{n})$, and (2) the normal component of $\mu\mathbf{v}$ on ∂K. The normal component of the vector $\mu\mathbf{v}$ is given by $(\mu\mathbf{v}\cdot\mathbf{n})\mathbf{n}$. Intuitively speaking, the vector $\mathbf{v}_b$ should carry those two pieces of orthogonal information by summing up $(\mu\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ and $\mathbf{n}\times(\mathbf{v}\times\mathbf{n})$:

$$
\mathbf{v}_b = (\mu\mathbf{v}\cdot\mathbf{n})\mathbf{n} + \mathbf{n}\times(\mathbf{v}\times\mathbf{n}).
$$

It is easy to check the following identity:

$$
(6.1)\qquad \mathbf{v}_b\times\mathbf{n} = (\mathbf{n}\times(\mathbf{v}\times\mathbf{n}))\times\mathbf{n} = \mathbf{v}\times\mathbf{n}.
$$

6.1. Weak divergence. Following [31], for any $\mathbf{v}\in V(K)$, we define the weak divergence of $\mu\mathbf{v}$, denoted by $\nabla_w\cdot(\mu\mathbf{v})$, as a bounded linear functional in the Sobolev space $H^1(K)$ such that

$$
\langle\nabla_w\cdot(\mu\mathbf{v}),\varphi\rangle_K = -(\mu\mathbf{v}_0,\nabla\varphi)_K + \langle\mathbf{v}_b\cdot\mathbf{n},\varphi\rangle_{\partial K},\qquad \forall\,\varphi\in H^1(K).
$$

The discrete weak divergence of $\mu\mathbf{v}$, denoted by $\nabla_{w,r,K}\cdot(\mu\mathbf{v})$, is defined as the unique polynomial in $P_r(K)$, satisfying

$$(\nabla_{w,r,K}\cdot(\mu\mathbf{v}),\varphi)_K = -(\mu\mathbf{v}_0,\nabla\varphi)_K + \langle \mathbf{v}_b\cdot\mathbf{n},\varphi\rangle_{\partial K}, \quad \forall\ \varphi\in P_r(K). \tag{6.2}$$

Assume that $\mathbf{v}_0$ is sufficiently smooth such that $\nabla\cdot(\mu\mathbf{v}_0)\in L^2(K)$. By applying the integration by parts to the first term on the right-hand side of (6.2), we have

$$(\nabla_{w,r,K}\cdot(\mu\mathbf{v}),\varphi)_K = (\nabla\cdot(\mu\mathbf{v}_0),\varphi)_K + \langle(\mathbf{v}_b-\mu\mathbf{v}_0)\cdot\mathbf{n},\varphi\rangle_{\partial K}, \tag{6.3}$$

for any $\varphi\in[P_r(K)]^3$.

6.2. Weak curl. The weak curl of $\mathbf{v}\in V(K)$ (see [20]), denoted by $\nabla_w\times\mathbf{v}$, is defined as a bounded linear functional in the Sobolev space $[H^1(K)]^3$, such that

$$\langle\nabla_w\times\mathbf{v},\varphi\rangle_K = (\mathbf{v}_0,\nabla\times\varphi)_K - \langle\mathbf{v}_b\times\mathbf{n},\varphi\rangle_{\partial K}, \quad \forall\ \varphi\in[H^1(K)]^3.$$

The discrete weak curl of $\mathbf{v}\in V(K)$, denoted by $\nabla_{w,r,K}\times\mathbf{v}$, is defined as the unique polynomial in $[P_r(K)]^3$, satisfying

$$(\nabla_{w,r,K}\times\mathbf{v},\varphi)_K = (\mathbf{v}_0,\nabla\times\varphi)_K - \langle\mathbf{v}_b\times\mathbf{n},\varphi\rangle_{\partial K}, \quad \forall\varphi\in[P_r(K)]^3. \tag{6.4}$$

For sufficiently smooth $\mathbf{v}_0$ such that $\nabla\times\mathbf{v}_0\in[L^2(K)]^3$, by applying the integration by parts to the first term on the right-hand side of (6.4), we obtain

$$(\nabla_{w,r,K}\times\mathbf{v},\varphi)_K = (\nabla\times\mathbf{v}_0,\varphi)_K - \langle(\mathbf{v}_b-\mathbf{v}_0)\times\mathbf{n},\varphi\rangle_{\partial K}, \tag{6.5}$$

for any $\varphi\in[P_r(K)]^3$.

7. Weak Galerkin Discretizations. A polyhedral partition of Ω is a family of polyhedra $\{T_j:\ j=1,2,\dots\}$ such that the two following conditions are satisfied: (1) the union of all polyhedra T_j is the domain Ω, and (2) the intersection of any two polyhedra, $T_j\cap T_i$ $(i\neq j)$, is either empty or a common face of T_j and T_i. Each partition cell T_j is called an element. A polyhedral partition with finite number of elements is called a finite element partition of Ω. Assume that $\{T_j\}_{j=1,\dots,N}$ is a finite element partition of Ω that is shape-regular according to [30]. Denote by $h_T=\mathrm{diam}(T)$ the diameter of the cell/element T, and $h=\max_T h_T$ the meshsize of the partition $\mathcal{T}_h=\{T_j\}_{j=1,\dots,N}$. Denote by $\mathcal{E}_h$ the set of all faces in $\mathcal{T}_h$ so that each $e\in\mathcal{E}_h$ is either on the boundary of Ω or shared by two elements. Denote by $\mathcal{E}_h^0=\mathcal{E}_h\setminus\partial\Omega$ the set of all interior faces in $\mathcal{E}_h$. By definition, for each interior face $e\in\mathcal{E}_h^0$, there are two elements T_j and T_i, $i\neq j$, such that $e=T_j\cap T_i$.

Let $k\geq 1$ be any integer. For each element $T\in\mathcal{T}_h$, define the local finite element space as

$$\mathbf{V}(k,T)=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}:\mathbf{v}_0\in[P_k(T)]^3,\mathbf{v}_b\in[P_k(e)]^3,\ e\in(\partial T\cap\mathcal{E}_h)\}.$$

The global weak finite element space for the vector-component is given by

$$\mathbf{V}_h=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}:\ \mathbf{v}|_T\in\mathbf{V}(k,T),\ \mathbf{v}_b|_{\partial T_1\cap e}=\mathbf{v}_b|_{\partial T_2\cap e},\ T\in\mathcal{T}_h, e\in\mathcal{E}_h^0\}, \tag{7.1}$$

where $\mathbf{v}_b|_{\partial T_j\cap e}$ is the value of $\mathbf{v}_b$ on the face e as seen from the element $T_j,\ j=1,2$. The finite element space for the Lagrange multiplier p is defined as

$$W_h = \{q:\ q\in L^2(\Omega),\ q|_T\in P_{k-1}(T), T\in\mathcal{T}_h\}.$$

The discrete weak divergence $(\nabla_{w,k-1}\cdot)$ and the discrete weak curl $(\nabla_{w,k-1}\times)$ can be computed by using (6.2) and (6.4) on each element; i.e.,

$$\begin{aligned}(\nabla_{w,k-1}\cdot(\mu\mathbf{v}))|_T =&\nabla_{w,k-1,T}\cdot(\mu\mathbf{v}|_T), \qquad \mathbf{v}\in\mathbf{V}_h,\\ (\nabla_{w,k-1}\times\mathbf{v})|_T =&\nabla_{w,k-1,T}\times(\mathbf{v}|_T), \qquad \mathbf{v}\in\mathbf{V}_h.\end{aligned}$$

For simplicity of notation, we shall drop the subscript $k-1$ from the notations $(\nabla_{w,k-1}\cdot)$ and $(\nabla_{w,k-1}\times)$ from now on.

Introduce the following bilinear forms

$$a(\mathbf{v},\mathbf{w}) =(\kappa\nabla_w\times\mathbf{v},\nabla_w\times\mathbf{w})_h + s(\mathbf{v},\mathbf{w}), \tag{7.2}$$

$$b(\mathbf{v},q) =(\nabla_w\cdot(\mu\mathbf{v}),q)_h, \tag{7.3}$$

where

$$(\kappa\nabla_w\times\mathbf{v},\nabla_w\times\mathbf{w})_h = \sum_{T\in\mathcal{T}_h}(\kappa\nabla_w\times\mathbf{v},\nabla_w\times\mathbf{w})_T,$$

$$(\nabla_w\cdot(\mu\mathbf{v}),q)_h = \sum_{T\in\mathcal{T}_h}(\nabla_w\cdot(\mu\mathbf{v}),q)_T,$$

$$\begin{aligned}s(\mathbf{v},\mathbf{w}) =& \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle(\mu\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n},(\mu\mathbf{w}_0-\mathbf{w}_b)\cdot\mathbf{n}\rangle_{\partial T}\\ &+\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle(\mathbf{v}_0-\mathbf{v}_b)\times\mathbf{n},(\mathbf{w}_0-\mathbf{w}_b)\times\mathbf{n}\rangle_{\partial T}.\end{aligned} \tag{7.4}$$

We are now in a position to describe a finite element method for the model problem given in (5.1) and (5.2). Consider first the variational formulation (5.2). Observe that the test space is $H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div}_\mu;\Omega)$. The corresponding analogue in the weak Galerkin setting is the following weak finite element space

$$\mathbf{V}_h^0 = \{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in\mathbf{V}_h:\ \mathbf{v}_b\times\mathbf{n}=0 \text{ on } \Gamma\}. \tag{7.5}$$

Algorithm 1. (weak Galerkin for the problem (5.2)) *Find $\boldsymbol{u}_h=\{\boldsymbol{u}_0,\boldsymbol{u}_b\}\in\mathbf{V}_h$, $p_h\in W_h$, and $\lambda\in\mathbb{R}^m$ with $\boldsymbol{u}_b\times\mathbf{n}=\mathbf{Q}_b\xi$ on Γ such that*

$$a(\boldsymbol{u}_h,\boldsymbol{v})+b(\boldsymbol{v},p_h)+\sum_{i=1}^m\langle\boldsymbol{v}_b\cdot\mathbf{n}_i,\lambda_i\rangle_{\Gamma_i} = (\mathbf{g},\boldsymbol{v}_0), \qquad \forall\ \boldsymbol{v}=\{\boldsymbol{v}_0,\boldsymbol{v}_b\}\in\mathbf{V}_h^0, \tag{7.6}$$

$$b(\boldsymbol{u}_h,w)+\sum_{i=1}^m\langle\boldsymbol{u}_b\cdot\mathbf{n}_i,s_i\rangle_{\Gamma_i} = (f,w)+\sum_{i=1}^m\beta_i s_i, \qquad \forall\ w\in W_h,\ s\in\mathbb{R}^m. \tag{7.7}$$

Here $\mathbf{Q}_b\xi$ is the usual L^2-projection to $[P_k(e)]^3$ for each boundary face $e\in(\Gamma\cap\mathcal{E}_h)$.

The Lagrange multiplier λ can be eliminated from the formulation (7.6)-(7.7) if the test space $\mathbf{V}_h^0$ is replaced by

$$\mathbf{U}_h^0 = \{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in\mathbf{V}_h^0:\ \langle\mathbf{v}_b\cdot\mathbf{n}_i,1\rangle_{\Gamma_i}=0,\ i=1,\dots,m\}. \tag{7.8}$$

The following is the corresponding weak Galerkin finite element scheme.

ALGORITHM 2. (weak Galerkin for the problem (5.1)) *Find* $\boldsymbol{u}_h = \{\boldsymbol{u}_0, \boldsymbol{u}_b\} \in \mathbf{V}_h$ *and* $p_h \in W_h$ *with* $\boldsymbol{u}_b \times \mathbf{n} = \mathbf{Q}_b \xi$ *on* Γ *such that*

$$a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) = (\mathbf{g}, \boldsymbol{v}_0), \qquad \forall\ \boldsymbol{v} = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in \mathbf{U}_h^0, \tag{7.9}$$

$$b(\boldsymbol{u}_h, w) = (f, w), \qquad \forall\ w \in W_h, \tag{7.10}$$

$$\langle \boldsymbol{u}_b \cdot \mathbf{n}_i, 1\rangle_{\Gamma_i} = \beta_i, \qquad i = 1, \ldots, m. \tag{7.11}$$

Note that the Algorithms 1 and 2 are equivalent in the sense that both give the same numerical solution $\mathbf{u}_h$ and p_h. Thus, it is sufficient to develop a convergence theory for Algorithm 2 only.

8. Existence and Uniqueness. The goal of this section is to show that the WG Algorithm 2 has one and only one solution.

We first introduce a topology in the weak finite element space $\mathbf{V}_h$ by defining a semi-norm as follows

$$\begin{aligned} |\!|\!|\mathbf{v}|\!|\!|_1 = &\Big(\sum_{T\in\mathcal{T}_h} \|\nabla_w \times \mathbf{v}\|_T^2 + \|\nabla_w \cdot (\mu \mathbf{v})\|_T^2 \\ &+ h_T^{-1}\|\mu\mathbf{v}_0\cdot\mathbf{n} - \mathbf{v}_b\cdot\mathbf{n}\|_{\partial T}^2 + h_T^{-1}\|\mathbf{v}_0\times\mathbf{n} - \mathbf{v}_b\times\mathbf{n}\|_{\partial T}^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{8.1}$$

For convenience, we set

$$\begin{aligned} |\!|\!|\mathbf{v}|\!|\!|^2 := a(\mathbf{v},\mathbf{v}) = &\sum_{T\in\mathcal{T}_h} (\kappa\nabla_w\times\mathbf{v}, \nabla_w\times\mathbf{v})_T \\ &+ h_T^{-1}\|\mu\mathbf{v}_0\cdot\mathbf{n} - \mathbf{v}_b\cdot\mathbf{n}\|_{\partial T}^2 + h_T^{-1}\|\mathbf{v}_0\times\mathbf{n} - \mathbf{v}_b\times\mathbf{n}\|_{\partial T}^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{8.2}$$

In the finite element space W_h, we introduce a mesh-dependent norm

$$\|q\|_{W_h}^2 = h^2 \sum_{T\in\mathcal{T}_h} \|\nabla q\|_T^2 + h \sum_{e\in\mathcal{E}_h^0} \|[\![q]\!]_e\|_e^2 + h\sum_{i=0}^{m} \|q - \bar{q}_i\|_{\Gamma_i}^2, \tag{8.3}$$

where $[\![q]\!]_e$ stands for the jump of q on the interior face $e \in \mathcal{E}_h^0$, $\bar{q}_0 = 0$, and $\bar{q}_i$ is the average of q on the connected boundary component Γ_i, $i = 1, \ldots, m$.

LEMMA 8.1. *Assume that the domain* Ω *is connected. Then, the semi-norm* $|\!|\!|\cdot|\!|\!|_1$ *defined as in (8.1) defines a norm in the linear space* $\mathbf{U}_h^0$.

Proof. It suffices to verify the positivity property for $|\!|\!|\cdot|\!|\!|_1$. To this end, assume that $|\!|\!|\mathbf{v}|\!|\!|_1 = 0$ for some $\mathbf{v} \in \mathbf{U}_h^0$. Thus,

$$\nabla_w \times \mathbf{v} = 0, \qquad \text{in } T, \tag{8.4}$$

$$\nabla_w \cdot (\mu\mathbf{v}) = 0, \qquad \text{in } T, \tag{8.5}$$

$$\mu\mathbf{v}_0\cdot\mathbf{n} - \mathbf{v}_b\cdot\mathbf{n} = 0, \qquad \text{on } \partial T \tag{8.6}$$

$$\mathbf{v}_0\times\mathbf{n} - \mathbf{v}_b\times\mathbf{n} = 0, \qquad \text{on } \partial T. \tag{8.7}$$

Using (8.4), (8.7) and (6.5), for any $\varphi \in [P_{k-1}(T)]^3$, we have

$$\begin{aligned} 0 =&(\nabla_w \times \mathbf{v}, \varphi)_T \\ =&(\nabla \times \mathbf{v}_0, \varphi)_T - \langle (\mathbf{v}_b - \mathbf{v}_0) \times \mathbf{n}, \varphi \rangle_{\partial T} \\ =&(\nabla \times \mathbf{v}_0, \varphi)_T. \end{aligned}$$

It follows that $\nabla \times \mathbf{v}_0 = 0$ on each element $T \in \mathcal{T}_h$, which along with (8.7) implies $\mathbf{v}_0 \in H(\text{curl}; \Omega)$ and

$$\nabla \times \mathbf{v}_0 = 0, \qquad \text{in } \Omega. \tag{8.8}$$

Next, using (8.5), (8.6) and (6.3), for any $\varphi \in P_{k-1}(T)$, we have

$$\begin{aligned} 0 =&(\nabla_w \cdot (\mu \mathbf{v}), \varphi)_T \\ =&(\nabla \cdot (\mu \mathbf{v}_0), \varphi)_T + \langle (\mathbf{v}_b - \mu \mathbf{v}_0) \cdot \mathbf{n}, \varphi \rangle_{\partial T} \\ =&(\nabla \cdot (\mu \mathbf{v}_0), \varphi)_T. \end{aligned}$$

It follows that $\nabla \cdot (\mu \mathbf{v}_0) = 0$ on each element $T \in \mathcal{T}_h$, which, together with (8.6), gives rise to $\mu \mathbf{v}_0 \in H(\text{div}; \Omega)$ and

$$\nabla \cdot (\mu \mathbf{v}_0) = 0, \qquad \text{in } \Omega. \tag{8.9}$$

Combining (8.6)-(8.7) with the fact that $\mathbf{v} \in \mathbf{U}_h^0$ yields

$$\begin{aligned} \mathbf{v}_0 \times \mathbf{n} = 0, &\quad \text{on } \Gamma, \\ \langle \mu \mathbf{v}_0 \cdot \mathbf{n}_i, 1 \rangle_{\Gamma_i} = 0, &\quad i = 1, \ldots, m, \end{aligned}$$

which, along with (8.8) and (8.9), implies $\mathbf{v}_0 = 0$ and hence $\mathbf{v}_b = 0$; see the proof of Lemma 4.1 for details. ∎

THEOREM 8.2. *The weak Galerkin finite element algorithms 1 and 2 have one and only one solution.*

Proof. Since the Algorithms 1 and 2 are equivalent, then it suffices to deal with Algorithms 2. As the number of unknowns is the same as the number of equations in (7.9)-(7.11), the existence of solution is equivalent to the uniqueness.

Let $(\mathbf{u}_h^{(j)}; p_h^{(j)}) \in \mathbf{V}_h \times W_h,\ j = 1, 2$, be two solutions of (7.9)-(7.11), and set

$$\mathbf{z}_h = \mathbf{u}_h^{(1)} - \mathbf{u}_h^{(2)}, \quad \gamma_h = p_h^{(1)} - p_h^{(2)}.$$

It is clear that $(\mathbf{z}_h; \gamma_h) \in \mathbf{U}_h^0 \times W_h$ satisfies

$$a(\mathbf{z}_h, \mathbf{v}) + b(\mathbf{v}, \gamma_h) = 0, \qquad \forall\, \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \mathbf{U}_h^0, \tag{8.10}$$

$$b(\mathbf{z}_h, w) = 0, \qquad \forall\, w \in W_h. \tag{8.11}$$

By letting $\mathbf{v} = \mathbf{z}_h$ in (8.10) and $w = \gamma_h$ in (8.11), we obtain

$$a(\mathbf{z}_h, \mathbf{z}_h) = 0, \quad \nabla_w \cdot (\mu \mathbf{z}_h) = 0,$$

which leads to $|\!|\!| \mathbf{z}_h |\!|\!|_1 = 0$. It follows from Lemma 8.1 that $\mathbf{z}_h = 0$, and thus $\mathbf{u}_h^{(1)} \equiv \mathbf{u}_h^{(2)}$.

To show $\gamma_h = 0$, we use (8.10) and $\mathbf{z}_h = 0$ to obtain

$$b(\mathbf{v}, \gamma_h) = 0, \qquad \forall\ \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \mathbf{U}_h^0.$$

From the definition of weak divergence (6.2),

$$\begin{aligned} b(\mathbf{v}, \gamma_h) &= \sum_{T\in\mathcal{T}_h} (\nabla_w \cdot (\mu\mathbf{v}), \gamma_h)_T \\ &= \sum_{T\in\mathcal{T}_h} -(\mu\mathbf{v}_0, \nabla\gamma_h)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, \gamma_h\rangle_{\partial T} \\ &= -\sum_{T\in\mathcal{T}_h} (\mu\mathbf{v}_0, \nabla\gamma_h)_T + \sum_{e\in\mathcal{E}_h} \langle \mathbf{v}_b \cdot \mathbf{n}_e, [\![\gamma_h]\!]\rangle_e, \end{aligned} \tag{8.12}$$

where $\mathbf{n}_e$ is a prescribed orientation of e. By letting $\mathbf{v} = \{-h^2\nabla\gamma_h;\ h\delta_e[\![\gamma_h]\!]\mathbf{n}_e\}$ with $\delta_e = 0$ when $e \subset \Gamma_i,\ i = 1, \ldots, m$ and $\delta_e = 1$ otherwise, we see that $\mathbf{v} \in \mathbf{U}_h^0$. Substituting this into (8.12) yields

$$0 = b(\mathbf{v}, \gamma_h) = h^2 \sum_{T\in\mathcal{T}_h} (\mu\nabla\gamma_h, \nabla\gamma_h)_T + h \sum_{e\in\mathcal{E}_h^0\cup\Gamma_0} \|[\![\gamma_h]\!]\|_e^2.$$

It follows that $\gamma_h = 0$, and thus $p_h^{(1)} \equiv p_h^{(2)}$. This completes the proof of uniqueness. $\square$

9. Error Equations. Let Q_0 be the L^2 projection onto $[P_k(T)]^3,\ T \in \mathcal{T}_h$, and Q_b the L^2 projection onto $[P_k(e)]^3,\ e \in \partial T \cap \mathcal{E}_h$. Denote by Q_h the L^2 projection onto the weak finite element space $\mathbf{V}_h$ such that on each element $T \in \mathcal{T}_h$,

$$(Q_h\mathbf{u})|_T = \{Q_0\mathbf{u}, \mathbb{Q}_b\mathbf{u}\}, \tag{9.1}$$

where

$$\mathbb{Q}_b\mathbf{u} = Q_b(\mu\mathbf{u}\cdot\mathbf{n})\mathbf{n} + Q_b(\mathbf{n}\times(\mathbf{u}\times\mathbf{n})). \tag{9.2}$$

Note that $\mathbf{n}\times(\mathbf{u}\times\mathbf{n}) = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$ is the tangential component of the vector $\mathbf{u}$ on the boundary of the element. When $\mu = I$ is the identity matrix, $(\mu\mathbf{u}\cdot\mathbf{n})\mathbf{n}$ is the normal component of $\mathbf{u}$. In general, $(\mu\mathbf{u}\cdot\mathbf{n})\mathbf{n} + \mathbf{n}\times(\mathbf{u}\times\mathbf{n})$ is not a decomposition of the vector $\mathbf{u}$ restricted on ∂T.

Denote by $\mathcal{Q}_h$ and $\mathbf{Q}_h$ the L^2 projections onto $P_{k-1}(T)$ and $[P_{k-1}(T)]^3$, respectively.

Lemma 9.1. *[20, 31] The projection operators Q_h, $\boldsymbol{Q}_h$, and $\mathcal{Q}_h$ satisfy the following commutative identities:*

$$\nabla_w\cdot(\mu Q_h\boldsymbol{v}) = \mathcal{Q}_h\nabla\cdot(\mu\boldsymbol{v}), \qquad \boldsymbol{v}\in H(\mathrm{div}_\mu;\Omega), \tag{9.3}$$

$$\nabla_w\times(Q_h\boldsymbol{v}) = \boldsymbol{Q}_h(\nabla\times\boldsymbol{v}), \qquad \boldsymbol{v}\in H(\mathrm{curl};\Omega). \tag{9.4}$$

Proof. It suffices to verify (9.3) on each element $T \in \mathcal{T}_h$. To this end, using the

definition (6.2) for the discrete weak divergence and (9.2), we obtain

$$\begin{aligned}(\nabla_w \cdot (\mu Q_h \mathbf{v}), \varphi)_T &= -(\mu Q_0 \mathbf{v}, \nabla\varphi)_T + \langle(\mathbb{Q}_b \mathbf{v})\cdot\mathbf{n}, \varphi\rangle_{\partial T}\\ &= -(\mu Q_0 \mathbf{v}, \nabla\varphi)_T + \langle Q_b(\mu\mathbf{v}\cdot\mathbf{n}), \varphi\rangle_{\partial T}\\ &= -(\mu\mathbf{v}, \nabla\varphi)_T + \langle\mu\mathbf{v}\cdot\mathbf{n}, \varphi\rangle_{\partial T}\\ &= (\nabla\cdot(\mu\mathbf{v}), \varphi)_T\\ &= (\mathcal{Q}_h\nabla\cdot(\mu\mathbf{v}), \varphi)_T\end{aligned}$$

for all $\varphi \in P_{k-1}(T)$. Thus, the identity (9.3) holds true. A similar argument can be applied to verify (9.4). ∎

Let $(\mathbf{u}_h; p_h) = (\{\mathbf{u}_0, \mathbf{u}_b\}; p_h) \in \mathbf{V}_h \times W_h$ be the WG finite element solution arising from Algorithm 2, and $(\mathbf{u}; p)$ be the solution of the continuous problem (5.1) or (5.2). The error functions are given by

$$\mathbf{e}_h = \{\mathbf{e}_0, \mathbf{e}_b\} = \{Q_0\mathbf{u} - \mathbf{u}_0, Q_b\mathbf{u} - \mathbf{u}_b\}, \tag{9.5}$$

$$\epsilon_h = \mathcal{Q}_h p - p_h. \tag{9.6}$$

Lemma 9.2. *Assume that $(\boldsymbol{w}; \rho) \in H(\mathrm{curl}; \Omega) \times L^2(\Omega)$ is sufficiently smooth on each element $T \in \mathcal{T}_h$ satisfying*

$$\nabla \times (\kappa\nabla \times \boldsymbol{w}) - \mu\nabla\rho = \eta, \qquad \text{in } \Omega, \tag{9.7}$$

$$\rho|_{\Gamma_0} = 0, \quad \rho|_{\Gamma_i} = const, \ i = 1, \ldots, m. \tag{9.8}$$

Denote by $\mathcal{Q}_h\rho$ the L^2 projection of ρ in the finite element space W_h. Then,

$$(\kappa\nabla_w \times (Q_h\boldsymbol{w}), \nabla_w \times \boldsymbol{v})_h + (\nabla_w\cdot(\mu\boldsymbol{v}), \mathcal{Q}_h\rho)_h = (\eta, \boldsymbol{v}_0) + l_{\boldsymbol{w}}(\boldsymbol{v}) + \theta_\rho(\boldsymbol{v}), \tag{9.9}$$

for all $\boldsymbol{v} \in \mathbf{U}_h^0$. Here $l_{\boldsymbol{w}}(\boldsymbol{v})$ and $\theta_\rho(\boldsymbol{v})$ are two functionals in the linear space $\mathbf{V}_h$ given by

$$l_{\boldsymbol{w}}(\boldsymbol{v}) = \sum_{T\in\mathcal{T}_h} \langle(\boldsymbol{Q}_h - I)(\kappa\nabla\times\boldsymbol{w}), (\boldsymbol{v}_0 - \boldsymbol{v}_b)\times\boldsymbol{n}\rangle_{\partial T}, \tag{9.10}$$

$$\theta_\rho(\boldsymbol{v}) = \sum_{T\in\mathcal{T}_h} \langle\rho - \mathcal{Q}_h\rho, (\mu\boldsymbol{v}_0 - \boldsymbol{v}_b)\cdot\mathbf{n}\rangle_{\partial T}. \tag{9.11}$$

Proof. It follows from (6.5) with $\varphi = \kappa\nabla_w \times (Q_h\mathbf{w})$ that

$$\begin{aligned}&(\nabla_w\times\mathbf{v}, \kappa\nabla_w\times(Q_h\mathbf{w}))_T =\\ &\qquad(\nabla\times\mathbf{v}_0, \kappa\nabla_w\times(Q_h\mathbf{w}))_T - \langle(\mathbf{v}_b - \mathbf{v}_0)\times\mathbf{n}, \kappa\nabla_w\times(Q_h\mathbf{w})\rangle_{\partial T}.\end{aligned}$$

Using (9.4), the above equation can be rewritten as

$$\begin{aligned}&(\kappa\nabla_w\times(Q_h\mathbf{w}), \nabla_w\times\mathbf{v})_T =\\ &\qquad(\kappa\nabla\times\mathbf{w}, \nabla\times\mathbf{v}_0)_T + \langle\mathbf{Q}_h(\kappa\nabla\times\mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b)\times\mathbf{n}\rangle_{\partial T}.\end{aligned}$$

Applying the integration by parts to the first term on the right-hand side yields

$$\begin{aligned}&(\kappa\nabla_w\times(Q_h\mathbf{w}), \nabla_w\times\mathbf{v})_T\\ &\quad=(\nabla\times(\kappa\nabla\times\mathbf{w}), \mathbf{v}_0)_T - \langle\kappa\nabla\times\mathbf{w}, \mathbf{v}_0\times\mathbf{n}\rangle_{\partial T}\\ &\qquad+\langle\mathbf{Q}_h(\kappa\nabla\times\mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b)\times\mathbf{n}\rangle_{\partial T}\\ &\quad=(\nabla\times(\kappa\nabla\times\mathbf{w}), \mathbf{v}_0)_T - \langle\kappa\nabla\times\mathbf{w}, \mathbf{v}_b\times\mathbf{n}\rangle_{\partial T}\\ &\qquad+\langle(\mathbf{Q}_h - I)(\kappa\nabla\times\mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b)\times\mathbf{n}\rangle_{\partial T}.\end{aligned} \tag{9.12}$$

Using (6.3) with $\varphi = \mathcal{Q}_h\rho$ and the usual integration by parts, we obtain

$$
\begin{aligned}
&(\nabla_w \cdot (\mu\mathbf{v}), \mathcal{Q}_h\rho)_T \\
&= (\nabla \cdot (\mu\mathbf{v}_0), \mathcal{Q}_h\rho)_T + \langle(\mathbf{v}_b - \mu\mathbf{v}_0)\cdot\mathbf{n}, \mathcal{Q}_h\rho\rangle_{\partial T} \\
&= (\nabla \cdot (\mu\mathbf{v}_0), \rho)_T + \langle(\mathbf{v}_b - \mu\mathbf{v}_0)\cdot\mathbf{n}, \mathcal{Q}_h\rho\rangle_{\partial T} \\
&= -(\mu\mathbf{v}_0, \nabla\rho)_T + \langle\mu\mathbf{v}_0\cdot\mathbf{n}, \rho\rangle_{\partial T} + \langle(\mathbf{v}_b - \mu\mathbf{v}_0)\cdot\mathbf{n}, \mathcal{Q}_h\rho\rangle_{\partial T} \\
&= -(\mathbf{v}_0, \mu\nabla\rho)_T + \langle(\mathbf{v}_b - \mu\mathbf{v}_0)\cdot\mathbf{n}, \mathcal{Q}_h\rho - \rho\rangle_{\partial T} + \langle\mathbf{v}_b\cdot\mathbf{n}, \rho\rangle_{\partial T}.
\end{aligned} \tag{9.13}
$$

Summing (9.12) over all the elements $T \in \mathcal{T}_h$ yields

$$
\begin{aligned}
(\kappa\nabla_w\times(Q_h\mathbf{w}), \nabla_w\times\mathbf{v})_h = &\sum_{T\in\mathcal{T}_h}(\nabla\times(\kappa\nabla\times\mathbf{w}), \mathbf{v}_0)_T \\
&+ \sum_{T\in\mathcal{T}_h}\langle(\mathbf{Q}_h - I)(\kappa\nabla\times\mathbf{w}), (\mathbf{v}_0 - \mathbf{v}_b)\times\mathbf{n}\rangle_{\partial T},
\end{aligned} \tag{9.14}
$$

where we have used two properties: (1) the cancelation property for the boundary integrals on interior faces, and (2) the fact that $\mathbf{v}_b \times \mathbf{n} = 0$ on Γ. Similarly, summing (9.13) over all the elements $T \in \mathcal{T}_h$, we obtain

$$
\begin{aligned}
(\nabla_w\cdot(\mu\mathbf{v}), \mathcal{Q}_h\rho)_h = &-(\mathbf{v}_0, \mu\nabla\rho) + \sum_{T\in\mathcal{T}_h}\langle(\mu\mathbf{v}_0 - \mathbf{v}_b)\cdot\mathbf{n}, \rho - \mathcal{Q}_h\rho\rangle_{\partial T} \\
&+ \sum_{e\in\mathcal{E}_h\cap\Gamma}\langle\mathbf{v}_b\cdot\mathbf{n}, \rho\rangle_e.
\end{aligned} \tag{9.15}
$$

The third term on the right-hand side of (9.15) vanishes if $\mathbf{v} \in \mathbf{U}_h^0$ and ρ satisfies the boundary condition (9.8). Thus, the equation (9.9) holds true by adding up (9.14) and (9.15). This completes the proof of the lemma. ∎

Theorem 9.3. *Let $(\boldsymbol{u}; p)$ be the solution of the model problem (5.1) or (5.2) and $(\boldsymbol{u}_h; p_h)$ be its numerical solution arising from the WG finite element scheme (7.9)-(7.11). Let the error functions $\boldsymbol{e}_h$ and ϵ_h be defined by (9.5)-(9.6). Then, $\boldsymbol{e}_h \in \mathbf{U}_h^0$ and the following error equations hold true*

$$
a(\boldsymbol{e}_h, \boldsymbol{v}) + b(\boldsymbol{v}, \epsilon_h) = \varphi_{\boldsymbol{u},p}(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \mathbf{U}_h^0, \tag{9.16}
$$

$$
b(\boldsymbol{e}_h, q) = 0, \qquad \forall q \in W_h, \tag{9.17}
$$

where

$$
\varphi_{\boldsymbol{u},p}(\boldsymbol{v}) = l_{\boldsymbol{u}}(\boldsymbol{v}) + \theta_p(\boldsymbol{v}) + s(Q_h\boldsymbol{u}, \boldsymbol{v}). \tag{9.18}
$$

Proof. Consider the model problem (5.2), as (5.1) is equivalent to (5.2). Let $(\mathbf{u}; p; \lambda)$ be the solution of this model problem. It is not hard to see that the following holds true:

$$
\begin{aligned}
&\nabla\times(\kappa\nabla\times\mathbf{u}) - \mu\nabla p = \mathbf{g}, \qquad \text{in } \Omega, \\
&p|_{\Gamma_0} = 0, \ p|_{\Gamma_i} = -\lambda_i, \quad i = 1, \ldots, m.
\end{aligned}
$$

Thus, by Lemma 9.2, we have

$$
(\kappa\nabla_w\times(Q_h\mathbf{u}), \nabla_w\times\mathbf{v})_h + (\nabla_w\cdot(\mu\mathbf{v}), \mathcal{Q}_h p)_h = (\mathbf{g}, \mathbf{v}_0) + l_{\mathbf{u}}(\mathbf{v}) + \theta_p(\mathbf{v})
$$

for all $\mathbf{v} \in \mathbf{U}_h^0$. It follows that

$$a(Q_h\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \mathcal{Q}_h p) = (\mathbf{g}, \mathbf{v}_0) + l_{\mathbf{u}}(\mathbf{v}) + \theta_p(\mathbf{v}) + s(Q_h\mathbf{u}, \mathbf{v}). \tag{9.19}$$

Subtracting (7.9) from (9.19) gives the first error equation (9.16).

Next, from the second equation in (5.1) and the commutative relation (9.3),

$$(f, q) = (\nabla \cdot (\mu\mathbf{u}), q)_h = (\mathcal{Q}_h \nabla \cdot (\mu\mathbf{u}), q)_h = (\nabla_w \cdot (\mu Q_h\mathbf{u}), q)_h. \tag{9.20}$$

The difference of (9.20) and (7.10) yields the second error equation (9.17). This completes the proof. □

10. The *inf-sup* Condition. For any $q \in W_h$, define a finite element function $\mathbf{v}_q \in \mathbf{U}_h^0$ by $\mathbf{v}_q = \{-h^2\nabla q;\ h\mathbf{v}_{q,b}\}$:

$$\mathbf{v}_{q,b} = \begin{cases} [\![q]\!]\ \mathbf{n}_e, & \text{on } \ e \in \mathcal{E}_h^0, \\ (q - \bar{q}_i)\ \mathbf{n}_i, & \text{on } \ e \in \mathcal{E}_h \cap \Gamma_i,\ \ i = 0, 1, \ldots, m. \end{cases} \tag{10.1}$$

Recall that $[\![q]\!]$ is the jump of q on the corresponding face $e \in \mathcal{E}_h^0$, $\mathbf{n}_e$ is a prescribed orientation of e, $\mathbf{n}_i$ is the outward normal direction on the connected component Γ_i, $\bar{q}_0 = 0$, and $\bar{q}_i$ is the average of q on Γ_i, $i = 1, \ldots, m$.

For any $\mathbf{v} = \{\mathbf{v}_0; \mathbf{v}_b\} \in \mathbf{U}_h^0$, from the definition of weak divergence (6.2), we have

$$\begin{aligned} b(\mathbf{v}, q) &= \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot (\mu\mathbf{v}), q)_T \\ &= \sum_{T \in \mathcal{T}_h} -(\mu\mathbf{v}_0, \nabla q)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, q\rangle_{\partial T} \\ &= -\sum_{T \in \mathcal{T}_h} (\mu\mathbf{v}_0, \nabla q)_T + \sum_{e \in \mathcal{E}_h^0} \langle \mathbf{v}_b \cdot \mathbf{n}_e, [\![q]\!]\rangle_e + \sum_{i=0}^{m} \langle \mathbf{v}_b \cdot \mathbf{n}_i, q\rangle_{\Gamma_i}. \end{aligned}$$

Note that $\langle \mathbf{v}_b \cdot \mathbf{n}_i, 1\rangle_{\Gamma_i} = 0$ for $i = 1, \ldots, m$. Thus,

$$b(\mathbf{v}, q) = -\sum_{T \in \mathcal{T}_h} (\mu\mathbf{v}_0, \nabla q)_T + \sum_{e \in \mathcal{E}_h^0} \langle \mathbf{v}_b \cdot \mathbf{n}_e, [\![q]\!]\rangle_e + \sum_{i=0}^{m} \langle \mathbf{v}_b \cdot \mathbf{n}_i, q - \bar{q}_i\rangle_{\Gamma_i}. \tag{10.2}$$

Lemma 10.1. (inf-sup condition) *For any $q \in W_h$, there exists a finite element function $\boldsymbol{v}_q \in \mathbf{U}_h^0$ such that*

$$b(\boldsymbol{v}_q, q) = h^2 \sum_{T \in \mathcal{T}_h} (\mu\nabla q, \nabla q)_T + h \sum_{e \in \mathcal{E}_h^0} \|[\![q]\!]\|_e^2 + h \sum_{i=0}^{m} \|q - \bar{q}_i\|_{\Gamma_i}^2, \tag{10.3}$$

$$|\!|\!| \boldsymbol{v}_q |\!|\!| \lesssim \|q\|_{W_h}. \tag{10.4}$$

Proof. For any $q \in W_h$, define $\mathbf{v}_{q,b}$ by (10.1) and set $\mathbf{v}_q = \{-h^2\nabla q;\ h\mathbf{v}_{q,b}\}$. On the boundary Γ, the vector $\mathbf{v}_{q,b}$ is parallel to the normal director $\mathbf{n}$. Thus, we have $\mathbf{v}_{q,b} \times \mathbf{n} = 0$ on Γ. Moreover, on each connected component Γ_i, we have

$$\langle \mathbf{v}_{q,b} \cdot \mathbf{n}_i, 1\rangle_{\Gamma_i} = \int_{\Gamma_i} (q - \bar{q}_i) = 0,\ \ i = 1, 2, \ldots, m.$$

Thus, $\mathbf{v}_q \in \mathbf{U}_h^0$.

Now, by taking $\mathbf{v} = \mathbf{v}_q$ in (10.2), we obtain

$$b(\mathbf{v}_q, q) = h^2 \sum_{T\in\mathcal{T}_h} (\mu\nabla q, \nabla q)_T + h \sum_{e\in\mathcal{E}_h^0} \|[\![q]\!]\|_e^2 + h \sum_{i=0}^{m} \|q - \bar{q}_i\|_{\Gamma_i}^2,$$

which verifies the identity (10.3).

To derive (10.4), we consider the following decomposition

$$\mathbf{v}_q = \mathbf{v}_q^{(1)} + \mathbf{v}_q^{(2)},$$

where $\mathbf{v}_q^{(1)} = -\{h^2\nabla q;\ 0\}$ and $\mathbf{v}_q^{(2)} = \{0;\ h\mathbf{v}_{q,b}\}$. It suffices to establish (10.4) for $\mathbf{v}_q^{(1)}$ and $\mathbf{v}_q^{(2)}$ independently.

From the semi-norm definition (8.2), we have

$$\begin{aligned} |\!|\!|\mathbf{v}_q^{(1)}|\!|\!|^2 = & \sum_{T\in\mathcal{T}_h} (\kappa\nabla_w \times \mathbf{v}_q^{(1)}, \nabla_w \times \mathbf{v}_q^{(1)})_T \\ & + h_T^{-1}\|h^2\mu\nabla q\cdot\mathbf{n}\|_{\partial T}^2 + h_T^{-1}\|h^2\nabla q \times \mathbf{n}\|_{\partial T}^2. \end{aligned} \tag{10.5}$$

The definition (6.4) for the discrete weak curl implies

$$(\nabla_w \times \mathbf{v}_q^{(1)}, \varphi)_T = -h^2(\nabla q, \nabla\times\varphi)_T, \qquad \forall\ \varphi \in [P_{k-1}(T)]^3.$$

It follows from the inverse inequality that

$$\|\nabla_w \times \mathbf{v}_q^{(1)}\|_T \lesssim h\|\nabla q\|_T.$$

Substituting the above into (10.5) and then using the trace inequality (11.3) yields

$$|\!|\!|\mathbf{v}_q^{(1)}|\!|\!|^2 \lesssim h^2\|\nabla q\|_T^2,$$

which verifies the estimate (10.4) for $\mathbf{v}_q^{(1)}$.

For $\mathbf{v}_q^{(2)}$, we again use the semi-norm definition (8.2) to obtain

$$|\!|\!|\mathbf{v}_q^{(2)}|\!|\!|^2 = \sum_{T\in\mathcal{T}_h} (\kappa\nabla_w\times\mathbf{v}_q^{(2)}, \nabla_w\times\mathbf{v}_q^{(2)})_T + h_T^{-1}\|h\mathbf{v}_{q,b}\cdot\mathbf{n}\|_{\partial T}^2 + h_T^{-1}\|h\mathbf{v}_{q,b}\times\mathbf{n}\|_{\partial T}^2. \tag{10.6}$$

Since $\mathbf{v}_{q,b}$ is parallel to $\mathbf{n}$, then $\mathbf{v}_{q,b} \times \mathbf{n} = 0$ on ∂T. In addition, the definition (6.4) for the discrete weak curl implies $\nabla_w \times \mathbf{v}_q^{(2)} = 0$ as

$$(\nabla_w \times \mathbf{v}_q^{(2)}, \varphi)_T = (0, \nabla\times\varphi)_T - h\langle \mathbf{v}_{q,b} \times \mathbf{n}, \varphi\rangle_{\partial T} = 0, \quad \forall\ \varphi \in [P_{k-1}(T)]^3.$$

Thus, it follows from (10.6) and (10.1) that

$$|\!|\!|\mathbf{v}_q^{(2)}|\!|\!|^2 \lesssim h \left(\sum_{e\in\mathcal{E}_h^0} \|[\![q]\!]\|_e^2 + \sum_{i=0}^{m} \|q - \bar{q}_i\|_{\Gamma_i}^2 \right),$$

which verifies the estimate (10.4) for $\mathbf{v}_q^{(2)}$. This completes the proof of the lemma. □

11. Error Analysis. Based on the error equations shown as in Theorem 9.3 and the *inf-sup* condition in the previous section , we shall derive an estimate for the error terms $\mathbf{e}_h$ and ϵ_h taken as the difference of the WG finite element solution and the L^2 projection of the exact solution.

11.1. Some technical inequalities. Assume that the finite element partition $\mathcal{T}_h$ of Ω is shape regular as defined in [30]. Let $T \in \mathcal{T}_h$ be an element with e as a face. The trace inequality holds true:

$$\|\psi\|_e^2 \lesssim \left(h_T^{-1}\|\psi\|_T^2 + h_T\|\nabla\psi\|_T^2\right), \qquad \forall\ \psi \in H^1(T). \tag{11.1}$$

If ϕ is a polynomial, the inverse inequality holds true:

$$\|\nabla\phi\|_T \lesssim h_T^{-1}\|\phi\|_T. \tag{11.2}$$

From (11.1) and (11.2), we have

$$\|\phi\|_e^2 \lesssim h_T^{-1}\|\phi\|_T^2. \tag{11.3}$$

Lemma 11.1. *[30] Let $k \geq 1$ be the order of the WG finite elements, and $1 \leq r \leq k$. Let $\boldsymbol{w} \in [H^{r+1}(\Omega)]^3$, $\rho \in H^r(\Omega)$, and $0 \leq m \leq 1$. There holds*

$$\sum_{T\in\mathcal{T}_h} h_T^{2m}\|\boldsymbol{w} - Q_0\boldsymbol{w}\|_{T,m}^2 \lesssim h^{2(r+1)}\|\boldsymbol{w}\|_{r+1}^2, \tag{11.4}$$

$$\sum_{T\in\mathcal{T}_h} h_T^{2m}\|\nabla\times\boldsymbol{w} - \boldsymbol{Q}_h(\nabla\times\boldsymbol{w})\|_{T,m}^2 \lesssim h^{2r}\|\boldsymbol{w}\|_{r+1}^2, \tag{11.5}$$

$$\sum_{T\in\mathcal{T}_h} h_T^{2m}\|\rho - \mathcal{Q}_h\rho\|_{T,m}^2 \lesssim h^{2r}\|\rho\|_r^2. \tag{11.6}$$

In the WG finite element space $\mathbf{V}_h$, we introduce a semi-norm as follows

$$|\mathbf{v}|_{1,h} = \left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\mathbf{v}_0 - \mathbf{v}_b)\times\mathbf{n}\|_{\partial T}^2 + h_T^{-1}\|(\mu\mathbf{v}_0 - \mathbf{v}_b)\cdot\mathbf{n}\|_{\partial T}^2\right)^{\frac{1}{2}}. \tag{11.7}$$

Lemma 11.2. *Assume that the finite element partition $\mathcal{T}_h$ of Ω is shape regular as defined in [30] and $1 \leq r \leq k$. Let $\boldsymbol{w} \in [H^{r+1}(\Omega)]^3$ and $\rho \in H^r(\Omega)$. Then, we have*

$$|s(Q_h\boldsymbol{w}, \boldsymbol{v})| \lesssim h^r\|\boldsymbol{w}\|_{r+1}\ |\boldsymbol{v}|_{1,h}, \tag{11.8}$$

$$|l_{\boldsymbol{w}}(\boldsymbol{v})| \lesssim h^r\|\boldsymbol{w}\|_{r+1}\ |\boldsymbol{v}|_{1,h}, \tag{11.9}$$

$$|\theta_\rho(\boldsymbol{v})| \lesssim h^r\|\rho\|_r\ |\boldsymbol{v}|_{1,h}, \tag{11.10}$$

for any $\boldsymbol{v} \in \mathbf{V}_h$. Here $l_{\boldsymbol{w}}(\cdot)$ and $\theta_\rho(\cdot)$ are defined in (9.10) and (9.11).

Proof. Recall from (7.4) that the stability term can be decomposed into two parts:

$$s(\mathbf{v}, \mathbf{w}) = s_1(\mathbf{v}, \mathbf{w}) + s_2(\mathbf{v}, \mathbf{w}),$$

where

$$s_1(\mathbf{v}, \mathbf{w}) = \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle(\mathbf{v}_0 - \mathbf{v}_b)\times\mathbf{n}, (\mathbf{w}_0 - \mathbf{w}_b)\times\mathbf{n}\rangle_{\partial T}, \tag{11.11}$$

$$s_2(\mathbf{v}, \mathbf{w}) = \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle(\mu\mathbf{v}_0 - \mathbf{v}_b)\cdot\mathbf{n}, (\mu\mathbf{w}_0 - \mathbf{w}_b)\cdot\mathbf{n}\rangle_{\partial T}. \tag{11.12}$$

To prove (11.8), it suffices to derive the estimate (11.8) for $s_1(\cdot,\cdot)$ and $s_2(\cdot,\cdot)$ separately. To this end, we use the Cauchy-Schwarz inequality, the trace inequality (11.1) and the estimate (11.4) to obtain

$$\begin{aligned}
|s_1(Q_h\mathbf{w},\mathbf{v})| &= \Big|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_0\mathbf{w}\times\mathbf{n} - Q_b(\mathbf{w}\times\mathbf{n}), (\mathbf{v}_0-\mathbf{v}_b)\times\mathbf{n}\rangle_{\partial T}\Big| \\
&= \Big|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_0\mathbf{w}\times\mathbf{n} - \mathbf{w}\times\mathbf{n}, (\mathbf{v}_0-\mathbf{v}_b)\times\mathbf{n}\rangle_{\partial T}\Big| \\
&\le \Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\mathbf{w}-\mathbf{w}\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\mathbf{v}_0-\mathbf{v}_b)\times\mathbf{n}\|_{\partial T}^2\Big)^{\frac12} \\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|Q_0\mathbf{w}-\mathbf{w}\|_T^2 + |Q_0\mathbf{w}-\mathbf{w}|_{1,T}^2\Big)^{\frac12}\ |\mathbf{v}|_{1,h} \\
&\lesssim h^r\|\mathbf{w}\|_{r+1}\ |\mathbf{v}|_{1,h}.
\end{aligned}$$

To derive (11.8) for $s_2(\cdot,\cdot)$, we again use the Cauchy-Schwarz inequality, the trace inequality (11.1) and the estimate (11.4) to obtain

$$\begin{aligned}
|s_2(Q_h\mathbf{w},\mathbf{v})| &= \Big|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle \mu Q_0\mathbf{w}\cdot\mathbf{n} - Q_b(\mu\mathbf{w}\cdot\mathbf{n}), (\mu\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\rangle_{\partial T}\Big| \\
&= \Big|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_0(\mu\mathbf{w})\cdot\mathbf{n} - \mu\mathbf{w}\cdot\mathbf{n}, (\mu\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\rangle_{\partial T}\Big| \\
&\le \Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mu(Q_0\mathbf{w}-\mathbf{w})\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\mu\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\|_{\partial T}^2\Big)^{\frac12} \\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|Q_0\mathbf{w}-\mathbf{w}\|_T^2 + |Q_0\mathbf{w}-\mathbf{w}|_{1,T}^2\Big)^{\frac12}\ |\mathbf{v}|_{1,h} \\
&\lesssim h^r\|\mathbf{w}\|_{r+1}\ |\mathbf{v}|_{1,h}.
\end{aligned}$$

As to (11.9), we use the Cauchy-Schwarz inequality, the trace inequality (11.1) and the estimate (11.5) to obtain

$$\begin{aligned}
|l_{\mathbf{w}}(\mathbf{v})| &= \left|\sum_{T\in\mathcal{T}_h}\langle(\mathbf{Q}_h-I)(\kappa\nabla\times\mathbf{w}), (\mathbf{v}_0-\mathbf{v}_b)\times\mathbf{n}\rangle_{\partial T}\right| \\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T\|(\mathbf{Q}_h-I)(\nabla\times\mathbf{w})\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\mathbf{v}_0-\mathbf{v}_b)\times\mathbf{n}\|_{\partial T}^2\Big)^{\frac12} \\
&\lesssim h^r\|\mathbf{w}\|_{r+1}\ |\mathbf{v}|_{1,h}.
\end{aligned}$$

Finally, we use the Cauchy-Schwarz inequality, the trace inequality (11.1) and the estimate (11.6) to obtain

$$\begin{aligned}
|\theta_\rho(\mathbf{v})| &= \left|\sum_{T\in\mathcal{T}_h}\langle\rho-\mathcal{Q}_h\rho, (\mu\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\rangle_{\partial T}\right| \\
&\le \Big(\sum_{T\in\mathcal{T}_h} h_T\|\rho-\mathcal{Q}_h\rho\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\mu\mathbf{v}_0-\mathbf{v}_b)\cdot\mathbf{n}\|_{\partial T}^2\Big)^{\frac12} \\
&\lesssim h^r\|\rho\|_r\ |\mathbf{v}|_{1,h}.
\end{aligned}$$

This completes the proof of the lemma. □

11.2. Error estimates. Recall that $|||\cdot|||_1$ defines a norm in the finite element space $\mathbf{U}_h^0$. This norm can be regarded as a discrete $H_0(\text{curl}) \cap H(\text{div}_\mu)$-norm under which the error function $\mathbf{e}_h$ shall be measured.

THEOREM 11.3. *Assume that $k \geq 1$ be the order of the WG finite element scheme (7.9)-(7.11). Let $(\boldsymbol{u}; p) \in [H^{k+1}(\Omega)]^3 \times H^k(\Omega)$ be the solution of the model problem (5.1) and $(\boldsymbol{u}_h; p_h)$ be the WG finite element solution arising from (7.9)-(7.11). Then, we have*

$$|||Q_h\boldsymbol{u} - \boldsymbol{u}_h|||_1 + \|\mathcal{Q}_h p - p_h\|_{W_h} \lesssim h^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k). \tag{11.13}$$

Proof. From Theorem 9.3, the error functions $\mathbf{e}_h = Q_h\mathbf{u} - \mathbf{u}_h$ and $\epsilon_h = \mathcal{Q}_h p - p_h$ satisfy the equations (9.16)-(9.17). By letting $\mathbf{v} = \mathbf{e}_h$ in (9.16) and then using (9.17), we obtain

$$a(\mathbf{e}_h, \mathbf{e}_h) = \varphi_{\mathbf{u},p}(\mathbf{e}_h). \tag{11.14}$$

The right-hand side can be estimated by using Lemma 11.2 as follows

$$|\varphi_{\mathbf{u},p}(\mathbf{e}_h)| \lesssim h^k(\|\mathbf{u}\|_{k+1} + \|p\|_k)|\mathbf{e}_h|_{1,h}.$$

Substituting the above into (11.14) yields

$$a(\mathbf{e}_h, \mathbf{e}_h) \lesssim h^k(\|\mathbf{u}\|_{k+1} + \|p\|_k)|\mathbf{e}_h|_{1,h},$$

which, together with $|\mathbf{e}_h|_{1,h}^2 \leq a(\mathbf{e}_h, \mathbf{e}_h)$, leads to

$$a(\mathbf{e}_h, \mathbf{e}_h)^{1/2} \lesssim h^k(\|\mathbf{u}\|_{k+1} + \|p\|_k). \tag{11.15}$$

Next, it follows from the equation (9.17) that $\nabla_w \cdot (\mu\mathbf{e}_h) = 0$. Thus,

$$|||\mathbf{e}_h|||_1 = |||\mathbf{e}_h||| \lesssim a(\mathbf{e}_h, \mathbf{e}_h)^{1/2}$$

Combining the above inequality with (11.15) gives

$$|||\mathbf{e}_h|||_1 \lesssim h^k(\|\mathbf{u}\|_{k+1} + \|p\|_k). \tag{11.16}$$

The error function ϵ_h can be estimated by using the *inf-sup* condition derived in Lemma 10.1. To this end, from the equation (9.16), we have

$$b(\mathbf{v}, \epsilon_h) = \varphi_{\mathbf{u},p}(\mathbf{v}) - a(\mathbf{e}_h, \mathbf{v}). \tag{11.17}$$

By letting $\mathbf{v} = \mathbf{v}_{\epsilon_h}$ in (11.17), we arrive at

$$\|\epsilon_h\|_{W_h}^2 \lesssim |\varphi_{\mathbf{u},p}(\mathbf{v}_{\epsilon_h})| + |a(\mathbf{e}_h, \mathbf{v}_{\epsilon_h})|.$$

Using Lemma 11.2 and the error estimate (11.15) we obtain

$$\|\epsilon_h\|_{W_h}^2 \lesssim h^k(\|\mathbf{u}\|_{k+1} + \|p\|_k)|||\mathbf{v}_{\epsilon_h}|||,$$

which, together with (10.4), leads to

$$\|\epsilon_h\|_{W_h} \lesssim h^k(\|\mathbf{u}\|_{k+1} + \|p\|_k).$$

This completes the proof of the theorem. $\square$

The mesh-dependent norm $\|q\|_{W_h}$ in the finite element space W_h is a scaled discrete H^1 norm for piecewise smooth functions. For the lowest order WG element (i.e., piecewise linear for $\mathbf{u}$ and piecewise constant for p), the error estimate in Theorem 11.3 does not give any convergence for the approximation of p. However, it is possible to replace the $\|\cdot\|_{W_h}$ norm by the standard L^2 norm in the error estimate (11.13) if the solution of the following second order elliptic problem is H^2-regular:

$$\begin{aligned} -\nabla\cdot(\mu\nabla\Phi) &= \epsilon_h, && \text{in } \Omega, \\ \Phi|_{\Gamma_i} &= \alpha_i, && i=0,1,\ldots,m, \end{aligned} \tag{11.18}$$

where $\alpha_0 = 0$ and α_i is a set of constants such that $\langle\nabla\Phi\cdot\mathbf{n}_i, 1\rangle_{\Gamma_i} = 0$ for $i = 1,\ldots,m$. In fact, for the solution of (11.18), it can be seen that $\nabla\Phi \in \mathbb{Y}_\mu(\Omega)\cap H(\mathrm{div}_\mu;\Omega)$. Moreover, the projection $\mathbf{z} = Q_h(\nabla\Phi)$ (see (9.1) for its definition) is a finite element function in $\mathbf{U}_h^0$. The desired error estimate for ϵ_h in $L^2(\Omega)$ can then be obtained by taking $\mathbf{v} = \mathbf{z}$ in the error equation (9.16). Details are left to interested readers as an exercise.

11.3. L^2-error estimates. To derive an L^2 error estimate for $\mathbf{e}_h$, we consider the dual problem of seeking $\boldsymbol{\psi} \in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div}_\mu;\Omega)$ and $\tau \in L^2(\Omega)$ such that

$$\begin{aligned} (\kappa\nabla\times\boldsymbol{\psi}, \nabla\times\mathbf{v}) + (\nabla\cdot(\mu\mathbf{v}), \tau) &= (\mathbf{e}_0, \mathbf{v}), && \forall\, \mathbf{v} \in \mathbb{Y}_\mu(\Omega)\cap H(\mathrm{div}_\mu;\Omega), \\ (\nabla\cdot(\mu\boldsymbol{\psi}), w) &= 0, && \forall\, w \in L^2(\Omega), \\ \langle\mu\boldsymbol{\psi}\cdot\mathbf{n}_i, 1\rangle_{\Gamma_i} &= 0, && i = 1,\ldots,m. \end{aligned} \tag{11.19}$$

Assume that the dual problem (11.19) has the $[H^2(\Omega)]^3\times H^1(\Omega)$-regularity property in the sense that the solution $(\boldsymbol{\psi};\tau) \in [H^2(\Omega)]^3\times H^1(\Omega)$ and satisfies the following a priori estimate:

$$\|\boldsymbol{\psi}\|_2 + \|\tau\|_1 \lesssim \|\mathbf{e}_0\|_0. \tag{11.20}$$

By using a Lagrange multiplier $\gamma = (\gamma_1,\ldots,\gamma_m) \in \mathbb{R}^m$, the dual problem (11.19) can be rewritten in an equivalent form as follows. Find $\boldsymbol{\psi} \in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div}_\mu;\Omega)$, $\tau \in L^2(\Omega)$, and $\gamma \in \mathbb{R}^m$ such that

$$\left\{ \begin{aligned} (\kappa\nabla\times\boldsymbol{\psi}, \nabla\times\mathbf{v}) + (\nabla\cdot(\mu\mathbf{v}), \tau) + \sum_{i=1}^m \langle\mu\mathbf{v}\cdot\mathbf{n}_i, \gamma_i\rangle_{\Gamma_i} &= (\mathbf{e}_0, \mathbf{v}), \\ (\nabla\cdot(\mu\boldsymbol{\psi}), w) + \sum_{i=1}^m \langle\mu\boldsymbol{\psi}\cdot\mathbf{n}_i, s_i\rangle_{\Gamma_i} &= 0, \end{aligned} \right. \tag{11.21}$$

for all $\mathbf{v} \in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div}_\mu;\Omega)$, $w \in L^2(\Omega)$, and $s \in \mathbb{R}^m$.

THEOREM 11.4. *Let $k \geq 1$ be the order of the WG scheme (7.9)-(7.11). Let $(\boldsymbol{u};p) \in [H^{k+1}(\Omega)]^3\times H^k(\Omega)$ and $(\boldsymbol{u}_h;p_h) \in \mathbf{V}_h\times W_h$ be the solutions of the problem (5.1) and (7.9)-(7.11), respectively. Then, the following estimate holds true*

$$\|Q_0\boldsymbol{u} - \boldsymbol{u}_0\| \lesssim h^{k+1}\big(\|\boldsymbol{u}\|_{k+1} + \|p\|_k\big). \tag{11.22}$$

Proof. From the first equation of (11.21), we see that the the equation (9.7) is satisfied by $(\mathbf{w};\rho) = (\boldsymbol{\psi};\tau)$ with $\eta = \mathbf{e}_0$. In addition, the boundary condition (9.8) is verified by

$$\tau|_{\Gamma_0} = 0,\ \tau|_{\Gamma_i} = -\gamma_i,\ i = 1,\ldots,m.$$

Thus, by using (9.9) in Lemma 9.2 with $\mathbf{v} = \mathbf{e}_h \in \mathbf{U}_h^0$, we obtain

$$\begin{aligned}\|Q_0\mathbf{u} - \mathbf{u}_0\|^2 =&(\kappa\nabla_w \times (Q_h\boldsymbol{\psi}), \nabla_w \times \mathbf{e}_h)_h \\ &+ (\nabla_w \cdot (\mu\mathbf{e}_h), \mathcal{Q}_h\tau)_h - \theta_\tau(\mathbf{e}_h) - l_{\boldsymbol{\psi}}(\mathbf{e}_h).\end{aligned} \tag{11.23}$$

Note that the error equation (9.17) implies $\nabla_w \cdot (\mu\mathbf{e}_h) = 0$ so that (11.23) can be simplified as

$$\|Q_0\mathbf{u} - \mathbf{u}_0\|^2 = a(Q_h\boldsymbol{\psi}, \mathbf{e}_h) - \varphi_{\boldsymbol{\psi},\tau}(\mathbf{e}_h), \tag{11.24}$$

where

$$\varphi_{\boldsymbol{\psi},\tau}(\mathbf{e}_h) = \theta_\tau(\mathbf{e}_h) + l_{\boldsymbol{\psi}}(\mathbf{e}_h) + s(Q_h\boldsymbol{\psi}, \mathbf{e}_h).$$

From (9.3) and the second equation of (11.19), we have

$$\begin{aligned}b(Q_h\boldsymbol{\psi}, \epsilon_h) &= (\nabla_w \cdot (\mu Q_h\boldsymbol{\psi}), \epsilon_h)_h \\ &= (\mathcal{Q}_h\nabla \cdot (\mu\boldsymbol{\psi}), \epsilon_h)_h = 0,\end{aligned}$$

which, together with (9.16) and (11.24), leads to

$$\begin{aligned}\|Q_0\mathbf{u} - \mathbf{u}_0\|^2 &= a(\mathbf{e}_h, Q_h\boldsymbol{\psi}) + b(Q_h\boldsymbol{\psi}, \epsilon_h) - \varphi_{\boldsymbol{\psi},\tau}(\mathbf{e}_h) \\ &= \varphi_{\mathbf{u},p}(Q_h\boldsymbol{\psi}) - \varphi_{\boldsymbol{\psi},\tau}(\mathbf{e}_h).\end{aligned} \tag{11.25}$$

The two terms on the righ-hand side of (11.25) can be estimated as follows. First, by letting $r = 1$, $\mathbf{v} = \mathbf{e}_h$, and $(\mathbf{w};\rho) = (\boldsymbol{\psi};\tau)$ in Lemma 11.2, we obtain

$$|\varphi_{\boldsymbol{\psi},\tau}(\mathbf{e}_h)| \lesssim h(\|\boldsymbol{\psi}\|_2 + \|\tau\|_1)\ |\mathbf{e}_h|_{1,h} \lesssim h\|\mathbf{e}_0\|\ \|\!|\mathbf{e}_h|\!\|, \tag{11.26}$$

where we have used (11.20) in the second inequality. Next, by letting $r = k$, $\mathbf{v} = Q_h\boldsymbol{\psi}$, and $(\mathbf{w};\rho) = (\mathbf{u};p)$ in Lemma 11.2, we arrive at

$$|\varphi_{\mathbf{u},p}(Q_h\boldsymbol{\psi})| \lesssim h^k(\|\mathbf{u}\|_{k+1} + \|p\|_k)\ |Q_h\boldsymbol{\psi}|_{1,h}. \tag{11.27}$$

To estimate $|Q_h\boldsymbol{\psi}|_{1,h}$, we recall from (11.7) and the definition (9.1) of Q_h that

$$|Q_h\boldsymbol{\psi}|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} h_T^{-1}\|(Q_0\boldsymbol{\psi} - \mathbb{Q}_b\boldsymbol{\psi}) \times \mathbf{n}\|_{\partial T}^2 + h_T^{-1}\|(\mu Q_0\boldsymbol{\psi} - \mathbb{Q}_b\boldsymbol{\psi}) \cdot \mathbf{n}\|_{\partial T}^2. \tag{11.28}$$

By using (9.2), we have

$$\begin{aligned}(\mathbb{Q}_b\boldsymbol{\psi}) \times \mathbf{n} &= Q_b(\mathbf{n} \times (\boldsymbol{\psi} \times \mathbf{n})) \times \mathbf{n} \\ &= Q_b\boldsymbol{\psi} \times \mathbf{n}, \\ (\mathbb{Q}_b\boldsymbol{\psi}) \cdot \mathbf{n} &= Q_b(\mu\boldsymbol{\psi} \cdot \mathbf{n}).\end{aligned}$$

Thus,

$$(Q_0\boldsymbol{\psi} - \mathbb{Q}_b\boldsymbol{\psi}) \times \mathbf{n} = (Q_0\boldsymbol{\psi} - Q_b\boldsymbol{\psi}) \times \mathbf{n},$$
$$(\mu Q_0\boldsymbol{\psi} - \mathbb{Q}_b\boldsymbol{\psi}) \cdot \mathbf{n} = \mu(Q_0\boldsymbol{\psi} - Q_b\boldsymbol{\psi}) \cdot \mathbf{n}.$$

Substituting the above identities into (11.28) yields

$$\begin{aligned}
|Q_h\boldsymbol{\psi}|_{1,h}^2 &= \sum_{T\in\mathcal{T}_h} h_T^{-1}\|(Q_0\boldsymbol{\psi} - Q_b\boldsymbol{\psi}) \times \mathbf{n}\|_{\partial T}^2 + h_T^{-1}\|\mu(Q_0\boldsymbol{\psi} - Q_b\boldsymbol{\psi}) \cdot \mathbf{n}\|_{\partial T}^2 \\
&\lesssim \sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\boldsymbol{\psi} - Q_b\boldsymbol{\psi}\|_{\partial T}^2 \\
&\leq \sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\boldsymbol{\psi} - \boldsymbol{\psi}\|_{\partial T}^2 \\
&\lesssim \sum_{T\in\mathcal{T}_h} (h_T^{-2}\|\boldsymbol{\psi} - Q_0\boldsymbol{\psi}\|_T^2 + \|\nabla(\boldsymbol{\psi} - Q_0\boldsymbol{\psi})\|_T^2) \\
&\lesssim \sum_{T\in\mathcal{T}_h} h_T^2\|\nabla^2\boldsymbol{\psi}\|_T^2 \\
&\lesssim h^2\|\boldsymbol{\psi}\|_2^2,
\end{aligned}$$

where we have used the L^2 property of Q_b in line 3, the trace inequality (11.1) in line 4, and the usual approximation property for the L^2 projection operator Q_0 in line 5. Inserting the above estimate into (11.27) and then using (11.20), we obtain

$$|\varphi_{\mathbf{u},p}(Q_h\boldsymbol{\psi})| \lesssim h^{k+1}(\|\mathbf{u}\|_{k+1} + \|p\|_k)\ \|\mathbf{e}_0\|. \tag{11.29}$$

Now, combining (11.25) with (11.26) and (11.29), we arrive at

$$\|Q_0\mathbf{u} - \mathbf{u}_0\|^2 \lesssim \Big(h\|\!|\mathbf{e}_h|\!\| + h^{k+1}(\|\mathbf{u}\|_{k+1} + \|p\|_k)\Big)\ \|\mathbf{e}_0\|.$$

It follows that

$$\|Q_0\mathbf{u} - \mathbf{u}_0\| \lesssim h\|\!|\mathbf{e}_h|\!\| + h^{k+1}(\|\mathbf{u}\|_{k+1} + \|p\|_k),$$

which, together with Theorem 11.3 and the fact that $\|\!|\mathbf{e}_h|\!\| \leq \|\!|\mathbf{e}_h|\!\|_1$, completes the proof of the theorem. ☐

The WG finite element solution $\mathbf{u}_h = \{\mathbf{u}_0; \mathbf{u}_b\}$ consists of two components on each element $T \in \mathcal{T}_h$: (1) the element interior value $\mathbf{u}_0$, and (2) the element boundary value $\mathbf{u}_b$. The rest of this section is devoted to an error analysis for $\mathbf{u}_b$. To this end, we introduce the following topology in the finite element space $\mathbf{V}_h$

$$\|\mathbf{v}_b\|_{\mathcal{E}_h} = \Big(\sum_{T\in\mathcal{T}_h} h_T \int_{\partial T} |\mathbf{v}_b|^2 ds\Big)^{\frac{1}{2}}, \qquad \mathbf{v} = \{\mathbf{v}_0; \mathbf{v}_b\} \in \mathbf{V}_h.$$

It is clear that the above defines an L^2-like norm for the face variable $\mathbf{v}_b$.

Theorem 11.5. *Let $k \geq 1$ be the order of the WG scheme (7.9)-(7.11). Let $(\boldsymbol{u}; p) \in [H^{k+1}(\Omega)]^3 \times H^k(\Omega)$ and $(\boldsymbol{u}_h; p_h) \in \mathbf{V}_h \times W_h$ be the solution of the problem (5.1) and (7.9)-(7.11), respectively. Then, we have*

$$\|\mathbb{Q}_b\boldsymbol{u} - \boldsymbol{u}_b\|_{\mathcal{E}_h} \lesssim h^{k+1}\big(\|\boldsymbol{u}\|_{k+1} + \|p\|_k\big). \tag{11.30}$$

Here $\mathbb{Q}_b \boldsymbol{u}$ is the projection of $\boldsymbol{u}$ on each face $e \in \mathcal{E}_h$ given by (9.2).

Proof. Note that $\mathbb{Q}_b \mathbf{u} - \mathbf{u}_b = \mathbf{e}_b$ is the error of the WG finite element solution on $\mathcal{E}_h$ – the set of all element faces in $\mathcal{T}_h$. It follows from the triangle inequality that on each ∂T

$$\begin{aligned}|\mathbf{e}_b|^2 &= |\mathbf{e}_b \cdot \mathbf{n}|^2 + |\mathbf{e}_b \times \mathbf{n}|^2 \\ &\le 2|(\mu \mathbf{e}_0 - \mathbf{e}_b) \cdot \mathbf{n}|^2 + 2|(\mathbf{e}_0 - \mathbf{e}_b) \times \mathbf{n}|^2 \\ &\quad + 2|\mu \mathbf{e}_0 \cdot \mathbf{n}|^2 + 2|\mathbf{e}_0 \times \mathbf{n}|^2.\end{aligned}$$

Summing over all the elements and then using (8.1) yields

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} h_T^{-1} \int_{\partial T} |\mathbf{e}_b|^2 ds &\lesssim |||\mathbf{e}_h|||_1^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1} \int_{\partial T} (|\mu \mathbf{e}_0 \cdot \mathbf{n}|^2 + |\mathbf{e}_0 \times \mathbf{n}|^2) ds \\ &\lesssim |||\mathbf{e}_h|||_1^2 + h^{-2} \|\mathbf{e}_0\|^2,\end{aligned}$$

where we have used the trace inequality (11.3) in the second line. The last inequality further leads to the following estimate:

$$\sum_{T\in\mathcal{T}_h} h_T \int_{\partial T} |\mathbf{e}_b|^2 ds \lesssim h^2 |||\mathbf{e}_h|||_1^2 + \|\mathbf{e}_0\|^2,$$

which, together with the error estimates (11.13) and (11.22), implies the desired estimate (11.30). □

REFERENCES

[1] G. Auchmuty and J.C. Alexander, *L^2 Well-posedness of planar div-curl systems*, Archive for Rational Mechanics and Analysis, 20 (2001), 160(2), pp. 91-134.

[2] I. Babuska, *The finite element method with Lagrange multipliers*, Numer. Math., 20 (1973), pp. 179-192.

[3] A. Bossavit, *Computational electromagnetism*, Academic Press Inc., San Diego, CA, 1998.

[4] J.H. Bramble and J.E. Pasciak, *A new approximation technique for div-curl systems*, Math. Comp., 73 (2003), pp. 1739-1762.

[5] S. Brenner, F. Li and L. Sung, *A locally divergence-free interior penalty method for two dimensional curl-curl problems*, SIAM J. Numer. Anal. 42 (2008), pp. 1190-1211.

[6] F. Brezzi, *On the existence, uniqueness, and approximation of saddle point problems arising from Lagrange multipliers*, RAIRO, 8 (1974), pp. 129-151.

[7] R. Brezzi and A. Buffa, *Innovative mimetic discretizations for electromagnetic problems*, J. Comput. Appl. Math., 234 (2010), pp. 1980-1987.

[8] C. L. Chang, *Finite element approximation for grad-div type of systems in the plane*, SIAM J. Numer. Anal., 29 (1992), pp. 590-601.

[9] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, 1978.

[10] D.M. Copeland, J. Gopalakrishnan, J.E. Pasciak, *A mixed method for axisymmetric div-curl systems*, Math. Comp. 77 (2008), pp. 1941-1965.

[11] M. Costabel, M. Dauge, and S. Nicaise, *Singularities of Maxwell interface problems*, M2AN Math. Model. Numer. Anal., 33(3) (1999), pp. 627-649.

[12] S. Delcourte, K. Domelevo, and P. Omnes, *A discrete duality finite volume approach to Hodge decomposition and div-curl problems on almost arbitrary two-dimensional meshes*, SIAM J. Numerical Analysis, 45 (2007), pp. 1142-1174.

[13] V. Girault and P-A Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer-Verlag Berlin Heidelberg, 1986.

[14] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numerica, 2002, pp. 237-339.

[15] P. Houston, I. Perugia and D. Schotzau, *hp-DGFEM for Maxwell's equations*, in Numerical Mathematics and Advanced Applications: ENUMATH 2001, F. Brezzi, A. Bua, S. Corsaro and A. Murli, eds., Springer-Verlag, Berlin, 2003, pp. 785-794.

[16] P. Houston, I. Perugia and D. Schotzau, *Mixed discontinuous Galerkin approximation of the Maxwell operator*, SIAM J. Numer. Anal., 42 (2004), pp. 434-459.

[17] K. Lipnikov, G. Manzini, F. Brezzi, and A. Buffa, *The mimetic finite difference method for the 3D magnetostatic field problems on polyhedral meshes*, J. Comput. Phys., 230 (2011), pp. 305-328.

[18] P. Monk, *Finite element methods for Maxwell's equations*, Oxford University Press, New York, 2003.

[19] L. Mu, J. Wang and X. Ye, *Weak Galerkin finite element methods on polytopal meshes*, arXiv:1204.3655v2, International Journal of Numerical Analysis and Modeling, Vol 12, Number 1, 2015, pp. 31-53.

[20] L. Mu, J. Wang, X. Ye and S. Zhang, *Weak Galerkin finite element method for the Maxwell equations*, arXiv:1312.2309, J. of Scientific Computing, to appear.

[21] J. Nédélec, *Mixed finite elements in R3*, Numer. Math., 35 (1980), pp. 315-341.

[22] R. A. Nicolaides, *Direct discretization of planar div-curl problems*, SIAM J. Numer. Anal., 29 (1992), pp. 32-56.

[23] R. A. Nicolaides and X. Wu, *Covolume solutions of three-dimensional div-curl equations*, SIAM J. Numer. Anal., 34 (1997), pp. 2195-2203.

[24] I. Perugia and D. Schotzau, *The hp-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations*, Math. Comp., 72 (2003), pp. 1179-1214.

[25] I. Perugia, D. Schotzau and P. Monk, *Stabilized interior penalty methods for the timeharmonic Maxwell equations*, Comput. Methods Appl. Mech. Engrg., 191 (2002), pp. 4675-4697.

[26] A.A. Rodriguez, E. Bertolazzi, R. Ghiloni, and A. Valli, *Construction of a finite element Basis of the first de Rham cohomology group and numerical solution of 3D magnetostatic problems*, SIAM J. Numer. Anal., 51(4), 2013, pp. 2380-2402.

[27] A.A. Rodriguez and A. Valli, *Eddy Current Approximation of Maxwell Equations*, Springer-Verlag Italia, Milan, 2010.

[28] J. Saranen, *On generalized harmonic fields in domains with anisotropic nonhomogeneous media*, J. Math. Anal. Appl., 88 (1982), pp. 104-115.

[29] C. Wang and J. Wang, *An efficient numerical scheme for the biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes*, arXiv:1309.5560, Computers and Mathematics with Applications, 68 (2014), pp. 2314-2330.

[30] J. Wang and X. Ye, *A weak Galerkin mixed finite element method for second-order elliptic problems*, arXiv:1202.3655v1, Math. Comp., Vol 83, Number 289, 2014, pp. 2101-2126.

[31] J. Wang and X. Ye, *A weak Galerkin finite element method for the Stokes equations*, arXiv:1302.2707.

[32] J. Wang and X. Ye, *A weak Galerkin finite element method for second-order elliptic problems*, arXiv:1104.2897, J. Comp. and Appl. Math, 241 (2013), pp. 103-115.